Question 6

The function f is defined by the power series

$$f(x) = -\frac{x}{2} + \frac{2x^2}{3} - \frac{3x^3}{4} + \cdots + \frac{(-1)^n nx^n}{n+1} + \cdots$$

for all real numbers x for which the series converges. The function g is defined by the power series

$$g(x) = 1 - \frac{x}{2!} + \frac{x^2}{4!} - \frac{x^3}{6!} + \cdots + \frac{(-1)^n x^n}{(2n)!} + \cdots$$

for all real numbers x for which the series converges.

(a) Find the interval of convergence of the power series for f. Justify your answer.

(b) The graph of $y = f(x) - g(x)$ passes through the point $(0, -1)$. Find $y'(0)$ and $y''(0)$. Determine whether y has a relative minimum, a relative maximum, or neither at $x = 0$. Give a reason for your answer.

(a) $$\left|\frac{(-1)^{n+1}(n+1)x^{n+1}}{n+2} \cdot \frac{n+1}{(-1)^n nx^n}\right| = \frac{(n+1)^2}{(n+2)(n)} \cdot |x|$$

$$\lim_{n\to\infty} \frac{(n+1)^2}{(n+2)(n)} \cdot |x| = |x|$$

The series converges when $-1 < x < 1$.

When $x = 1$, the series is $-\frac{1}{2} + \frac{2}{3} - \frac{3}{4} + \cdots$

This series does not converge, because the limit of the individual terms is not zero.

When $x = -1$, the series is $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \cdots$

This series does not converge, because the limit of the individual terms is not zero.

Thus, the interval of convergence is $-1 < x < 1$.

$5 : \begin{cases} 1 : \text{sets up ratio} \\ 1 : \text{computes limit of ratio} \\ 1 : \text{identifies radius of convergence} \\ 1 : \text{considers both endpoints} \\ 1 : \text{analysis/conclusion for both endpoints} \end{cases}$

(b) $f'(x) = -\frac{1}{2} + \frac{4}{3}x - \frac{9}{4}x^2 + \cdots$ and $f'(0) = -\frac{1}{2}$.

$g'(x) = -\frac{1}{2!} + \frac{2}{4!}x - \frac{3}{6!}x^2 + \cdots$ and $g'(0) = -\frac{1}{2}$.

$y'(0) = f'(0) - g'(0) = 0$

$f''(0) = \frac{4}{3}$ and $g''(0) = \frac{2}{4!} = \frac{1}{12}$.

Thus, $y''(0) = \frac{4}{3} - \frac{1}{12} > 0$.

Since $y'(0) = 0$ and $y''(0) > 0$, y has a relative minimum at $x = 0$.

$4 : \begin{cases} 1 : y'(0) \\ 1 : y''(0) \\ 1 : \text{conclusion} \\ 1 : \text{reasoning} \end{cases}$

Work for problem 6(a)

$$\lim_{n\to\infty}\left|\frac{\frac{(-1)^{n+1}(n+1)x^{n+1}}{n+2}}{\frac{(-1)^n\cdot n\cdot x^n}{n+1}}\right| \Rightarrow \lim_{n\to\infty}\frac{\cancel{(-1)^{n+1}}\cdot(n+1)\cdot x^{n+1}\cdot(n+1)}{\cancel{(-1)^n}\cdot n\cdot x^n\cdot(n+2)}$$

$$\Rightarrow \lim_{n\to\infty}|x|$$

$\Rightarrow R = 1$

$\Rightarrow \boxed{(-1, 1)}$

$(-1)^{2n}\frac{n}{n+1}$ diverges due to the n^{th} term test

$$\lim_{n\to\infty}\frac{(-1)^n\cdot n\cdot(-1)^n}{n+1}\neq 0$$

diverges due to the n^{th} term test

$$\lim_{n\to\infty}\frac{(-1)^n\cdot n\cdot 1^n}{n+1}\neq 0$$

Continue problem 6 on page 15.

NO CALCULATOR ALLOWED

Work for problem 6(b)

$g(x) = \cos x$

$$y' = \frac{(-1)^n \cdot n \cdot x^{n-1}}{(n+1)(n-1)} - \frac{(-1)^n \cdot x^{n-1}}{(2n)!\,(n+1)} \therefore y'(0) = 0$$

$f'(x) = -1/2 + \frac{4}{3}x \ldots \quad \therefore f'(0) = -1/2$

$g'(x) = -1/2! + \frac{2}{4!}x \ldots \quad g'(0) = -1/2 \quad \therefore y'(0) = \boxed{0}$

$f''(x) = 4/3$

$g''(x) = 1/12 \quad \therefore y''(0) = 4/3 - 1/12 = \frac{15}{12} = \boxed{\frac{5}{4}}$

has a relative min at $x=0$ because the ~~des~~ derivative shows there is a critical point at $x=0$ and the 2nd derivative shows a positive concavity meaning the function values are decreasing up till $x=0$ and increasing after therefore showing a relative minimum

STOP

END OF EXAM

THE FOLLOWING INSTRUCTIONS APPLY TO THE COVERS OF THE SECTION II BOOKLET.

- **MAKE SURE YOU HAVE COMPLETED THE IDENTIFICATION INFORMATION AS REQUESTED ON THE FRONT AND BACK COVERS OF THE SECTION II BOOKLET.**
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- **MAKE SURE YOU HAVE USED THE SAME SET OF AP NUMBER LABELS ON ALL AP EXAMS YOU HAVE TAKEN THIS YEAR.**

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Work for problem 6(a)

$$\frac{(n+1)\cancel{x^{(n+1)}}}{(n+1)+1} \cdot \frac{n+1}{n\cancel{x^n}} = \frac{(n^2+2n+1)x}{(n^2+n)} = 3x$$

$-1 \le 3x \le 1$

$$\boxed{-\frac{1}{3} \le x \le \frac{1}{3}}$$

$\frac{(-1)^n n(-\frac{1}{3})^n}{n+1} \to$ diverges

$\frac{(-1)^n n(\frac{1}{3})^n}{n+1} \to$ converges $\to$ alternating

* when $-\frac{1}{3} < x \le \frac{1}{3}$, the power series for f converges.

Continue problem 6 on page 15.

NO CALCULATOR ALLOWED

Work for problem 6(b)

$y' = f'(x) - g'(x)$

$y'(0) = -\frac{1}{2} + \frac{1}{2} = \boxed{0}$

$y'' = f''(x) - g''(x)$

$y''(0) = \frac{4}{3} - \frac{1}{12} =$

$\frac{16}{12} - \frac{1}{12} = \boxed{\frac{15}{12}}$

$f'(x) = -\frac{1}{2} + \frac{4x}{3} \ldots$

$f''(x) = \frac{4}{3}$

$g'(x) = -\frac{1}{2} + \frac{1}{12}x$

$g''(x) = \frac{1}{12}$

y has a relative minimum at x=0 because the derivative of y at x=0 is 0 and the Second derivative of y at x=0 is >0, meaning the graph of y is concave up at this point.

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Work for problem 6(a)

$$f(x) = \sum_{n=1}^{\infty} \frac{(-1)^n n x^n}{n+1}$$

$$L = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|$$

$$L = \lim_{n \to \infty} \left| \frac{(-1)^{n+1}(n+1)x^{n+1}}{n+2} \cdot \frac{n+1}{(-1)^n n x^n} \right|$$

$$L = \lim_{n \to \infty} \left| (-1)x \cdot \frac{(n+1)^2}{(n+2)(n)} \right|$$

$$L = x$$

$$L < 1$$

$$\boxed{x < 1}$$

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Continue problem 6 on page 15.

Work for problem 6(b)

$y = f(x) - g(x)$

$y'(x) = \left[-\frac{1}{2} + \frac{4x}{3} + \ldots \right] - \left[-\frac{1}{2} + \frac{2x}{4!} + \ldots \right]$

$y'(0) = -\frac{1}{2} - \left(-\frac{1}{2}\right)$

$y'(0) = 0$

$y''(x) = \left[\frac{4}{3} - \frac{18}{4}x + \ldots \right] - \left[\frac{2}{24} - \frac{6x}{6!} + \ldots \right]$

$y''(0) = \frac{4}{3} - \frac{1}{12} = \boxed{\frac{15}{12}}$

x ← − 0 + →

relative minimum

STOP

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Question 6

Overview

This problem dealt with power series. Students were given the power series expansions of two functions, f and g. In part (a) they were asked to find the interval of convergence of the power series for f. Part (b) dealt with the graph of $y = f(x) - g(x)$. Students had to know how to read or compute the values of the first and second derivatives of y at $x = 0$ from the series for f and g. They then needed to use this information to describe the nature of the critical point of y at $x = 0$.

Sample: 6A
Score: 9

The student earned all 9 points. In part (b) the student restarts the problem on the third line and earned all points.

Sample: 6B
Score: 6

The student earned 6 points: 2 points in part (a) and 4 points in part (b). In part (a) the student sets up an incorrect Ratio Test and does not compute a limit. The student earned the 2 points by stating the correct interval of convergence for the ratio and testing the endpoints. In part (b) the student calculates $y'(0)$ and $y''(0)$ and arrives at the correct conclusion with good reasoning.

Sample: 6C
Score: 4

The student earned 4 points: 1 point in part (a) and 3 points in part (b). In part (a) the student sets up the ratio correctly but does not identify the correct interval of convergence. In part (b) the student correctly calculates $y'(0)$ and $y''(0)$ and arrives at a correct conclusion. However, the student does not include a reason for the answer.