AP® CALCULUS BC
2007 SCORING GUIDELINES (Form B)

Question 5

Consider the differential equation $\frac{dy}{dx} = 3x + 2y + 1$.

(a) Find $\frac{d^2y}{dx^2}$ in terms of x and y.

(b) Find the values of the constants m, b, and r for which $y = mx + b + e^{rx}$ is a solution to the differential equation.

(c) Let $y = f(x)$ be a particular solution to the differential equation with the initial condition $f(0) = -2$. Use Euler's method, starting at $x = 0$ with a step size of $\frac{1}{2}$, to approximate $f(1)$. Show the work that leads to your answer.

(d) Let $y = g(x)$ be another solution to the differential equation with the initial condition $g(0) = k$, where k is a constant. Euler's method, starting at $x = 0$ with a step size of 1, gives the approximation $g(1) \approx 0$. Find the value of k.

(a) $\frac{d^2y}{dx^2} = 3 + 2\frac{dy}{dx} = 3 + 2(3x + 2y + 1) = 6x + 4y + 5$	$2 : \begin{cases} 1 : 3 + 2\frac{dy}{dx} \\ 1 : \text{answer} \end{cases}$
(b) If $y = mx + b + e^{rx}$ is a solution, then $m + re^{rx} = 3x + 2\left(mx + b + e^{rx}\right) + 1.$ If $r \neq 0$: $m = 2b + 1$, $r = 2$, $0 = 3 + 2m$, so $m = -\frac{3}{2}$, $r = 2$, and $b = -\frac{5}{4}$. OR If $r = 0$: $m = 2b + 3$, $r = 0$, $0 = 3 + 2m$, so $m = -\frac{3}{2}$, $r = 0$, $b = -\frac{9}{4}$.	$3 : \begin{cases} 1 : \frac{dy}{dx} = m + re^{rx} \\ 1 : \text{value for } r \\ 1 : \text{values for } m \text{ and } b \end{cases}$
(c) $f\left(\frac{1}{2}\right) \approx f(0) + f'(0) \cdot \frac{1}{2} = -2 + (-3) \cdot \frac{1}{2} = -\frac{7}{2}$ $f'\left(\frac{1}{2}\right) \approx 3\left(\frac{1}{2}\right) + 2\left(-\frac{7}{2}\right) + 1 = -\frac{9}{2}$ $f(1) \approx f\left(\frac{1}{2}\right) + f'\left(\frac{1}{2}\right) \cdot \frac{1}{2} = -\frac{7}{2} + \left(-\frac{9}{2}\right) \cdot \frac{1}{2} = -\frac{23}{4}$	$2 : \begin{cases} 1 : \text{Euler's method with 2 steps} \\ 1 : \text{Euler's approximation for } f(1) \end{cases}$
(d) $g'(0) = 3 \cdot 0 + 2 \cdot k + 1 = 2k + 1$ $g(1) \approx g(0) + g'(0) \cdot 1 = k + (2k + 1) = 3k + 1 = 0$ $k = -\frac{1}{3}$	$2 : \begin{cases} 1 : g(0) + g'(0) \cdot 1 \\ 1 : \text{value of } k \end{cases}$

NO CALCULATOR ALLOWED

BC5
5A₁

Work for problem 5(a)

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = 3 + 2 \cdot \frac{dy}{dx} = 3 + 2(3x+2y+1)$$

$$= 3 + 6x + 4y + 2 = \underline{\underline{6x+4y+5}}$$

Work for problem 5(b)

$y = mx + b + e^{rx}$ $\quad e^{rx} = y - mx - b$

$$\frac{dy}{dx} = m + re^{rx} = m + r(y - mx - b)$$

$$= -rmx + ry + (m - br) = 3x + 2y + 1$$

$-rm = 3$, $r = 2$, $m - br = 1$

$m = \frac{3}{-r} = -\frac{3}{2}$ $\quad br = m - 1 \Rightarrow b = \frac{m-1}{r} = \left(-\frac{3}{2} - 1\right) \cdot \frac{1}{2} = -\frac{5}{2} \cdot \frac{1}{2} = -\frac{5}{4}$

$\therefore m = -\frac{3}{2}$, $r = 2$, $b = -\frac{5}{4}$

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BC5
5A2

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Work for problem 5(c)

$f(\frac{1}{2}) = f(0+\frac{1}{2}) \approx f(0) + \frac{1}{2}\cdot f'(0) = -2 + \frac{1}{2}(3\cdot 0 + 2\cdot(-2)+1)$

$= -2 + \frac{1}{2}(-4+1) = -2 - \frac{3}{2} = -\frac{7}{2}$

$f(1) = f(\frac{1}{2}+\frac{1}{2}) \approx f(\frac{1}{2}) + \frac{1}{2}\cdot f'(\frac{1}{2}) = -\frac{7}{2} + \frac{1}{2}(3\cdot\frac{1}{2} + 2\cdot(-\frac{7}{2})+1)$

$= -\frac{7}{2} + \frac{1}{2}\left(\frac{3}{2} - 7 + 1\right) = -\frac{7}{2} + \frac{1}{2}\cdot(-\frac{9}{2}) = -\frac{7}{2} - \frac{9}{4}$

$= -\frac{23}{4}$ $\quad \therefore f(1) \approx -\frac{23}{4}$

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Work for problem 5(d)

$g(1) = g(0+1) \approx g(0) + 1\cdot g'(0) = k + 1\cdot(3\cdot 0 + 2\cdot k + 1)$

$= k + 2k + 1 = 3k+1 = 0 \Rightarrow 3k = -1$

$k = -\frac{1}{3}$ $\quad \therefore k = -\frac{1}{3}$

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Work for problem 5(a)

$$\frac{d^2y}{dx^2} = 3+2x\frac{dy}{dx}$$
$$= 3+2(3x+2y+1)$$
$$= 3+6x+4y+2$$
$$= 6x+4y+5$$

Work for problem 5(b)

$$\frac{dy}{dx} = m+re^{rx}$$
$$= 3x+2y+1$$

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Work for problem 5(c)

$f(0) = -2$

$f(0.5) = f(0) + f'(0) \times 0.5$

$= -2 + (-3) \times 0.5$

$= -3.5$

$f(1) = f(0.5) + f'(0.5) \times 0.5$

$= 3.5 + (-3) \times 0.5$

$= -5$

$\therefore f(1) \approx -5$

Work for problem 5(d)

$g(0) = k$

$g(0.5) = g(0) + g'(0) \times 1$

$= k + (2k+1)$

$= 0$

$3k + 1 = 0$

$\therefore k = -\frac{1}{3}$

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5C1

Work for problem 5(a)

$\frac{dy}{dx} = 3x + 2y + 1$

$\frac{d^2y}{dx^2} = 3 + 2\frac{dy}{dx} = 3 + 2\cdot(3x+2y+1)$

$= 3 + 6x + 4y + 2$

$= \underline{6x + 4y + 5}$

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Work for problem 5(b)

$dy = (3x + 2y + 1)dx \Rightarrow \int dy = \int(3x+2y+1)dx$

$y = \frac{3}{2}x^2 + 2xy + x$

$y - 2xy = y(1-2x) = \frac{3}{2}x^2 + x$

$y = \frac{3x^2+2x}{(1-2x)2} = \frac{3x^2+2x}{2-4x} = mx + b + e^{rx}$

$3x^2 + 2x = 2mx + 2b + 2e^{rx}$
$-4mx^2 - 4bx - 4xe^{rx}$

$2b + 2e^{rx} = 0$

$m = -\frac{4}{3}$

$-b = e^{rx}$

$2 = -\frac{8}{3} - 4b - 4e^{rx}$

$-4b - 4e^{rx} = \frac{14}{3}$

$2x = 2mx - 4bx - 4xe^{rx}$

Continue problem 5 on page 13.

NO CALCULATOR ALLOWED

BC
5C$_2$

Work for problem 5(c)

$Y_0 = -2 \quad x_0 = 0$

$x = 1 \quad y = ?$

$m = -\frac{4}{3}$ from 5(b)

$x_0 + \frac{1}{2} = \frac{1}{2} \quad Y_0 = -2$

$Y_1 = Y_0 + \frac{1}{2}(Y'_0) \quad Y'_1 = -\frac{4}{3}$

$= -2 + \frac{1}{2}(-\frac{4}{3})$

$= -\frac{8}{3}$

$-\frac{10}{3}$

$x_1 + \frac{1}{2} = 1 = x$

$Y_2 = Y_1 + \frac{1}{2}(-\frac{4}{3}) = -\frac{8}{3} - \frac{2}{3} = -\frac{10}{3}$

Work for problem 5(d)

$x_0 = 0 \quad Y_0 = k$

$x_0 + \frac{1}{2} = x_1 = \frac{1}{2}$

$x_1 + \frac{1}{2} = x_2 = 1$

$Y_1 = Y_0 + \frac{1}{2}(Y'_0) = k - \frac{2}{3}$

$Y_2 = (k - \frac{2}{3}) + \frac{1}{2} \cdot -\frac{4}{3}$

$= k - \frac{4}{3} = 0$

$k = \frac{4}{3}$

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Question 5

Sample: 5A
Score: 9

The student earned all 9 points.

Sample: 5B
Score: 6

The student earned 6 points: 2 points in part (a), 1 point in part (b), 1 point in part (c), and 2 points in part (d). The student presents correct work in parts (a) and (d). In part (b) the student only finds the correct $\frac{dy}{dx}$, and so the first point was earned. In part (c) the student earned the first point by the use of Euler's method with two steps to approximate $f(1)$. The student makes an error in calculating $f'\left(\frac{1}{2}\right)$, so the second point was not earned.

Sample: 5C
Score: 3

The student earned 3 points: 2 points in part (a), no points in part (b), 1 point in part (c), and no points in part (d). The student presents correct work in part (a). In part (b) the student does not find $\frac{dy}{dx}$. In part (c) the student earned the first point by the use of Euler's method with two steps to approximate $f(1)$. The student makes an error in calculating $f'\left(\frac{1}{2}\right)$, so the second point was not earned. In part (d) the student uses $\frac{1}{2}$ instead of 1 for Δx and makes computational errors, so no points were awarded.