AP® CALCULUS BC
2007 SCORING GUIDELINES (Form B)

Question 6

Let f be the function given by $f(x) = 6e^{-x/3}$ for all x.

(a) Find the first four nonzero terms and the general term for the Taylor series for f about $x = 0$.

(b) Let g be the function given by $g(x) = \int_0^x f(t)\,dt$. Find the first four nonzero terms and the general term for the Taylor series for g about $x = 0$.

(c) The function h satisfies $h(x) = k f'(ax)$ for all x, where a and k are constants. The Taylor series for h about $x = 0$ is given by

$$h(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots.$$

Find the values of a and k.

(a) $$f(x) = 6\left[1 - \frac{x}{3} + \frac{x^2}{2!3^2} - \frac{x^3}{3!3^3} + \cdots + \frac{(-1)^n x^n}{n!3^n} + \cdots\right]$$

$$= 6 - 2x + \frac{x^2}{3} - \frac{x^3}{27} + \cdots + \frac{6(-1)^n x^n}{n!3^n} + \cdots$$

$3 : \begin{cases} 1 : \text{two of } 6, -2x, \dfrac{x^2}{3}, -\dfrac{x^3}{27} \\ 1 : \text{remaining terms} \\ 1 : \text{general term} \end{cases}$

$\langle -1 \rangle$ missing factor of 6

(b) $g(0) = 0$ and $g'(x) = f(x)$, so

$$g(x) = 6\left[x - \frac{x^2}{6} + \frac{x^3}{3!3^2} - \frac{x^4}{4!3^3} + \cdots + \frac{(-1)^n x^{n+1}}{(n+1)!3^n} + \cdots\right]$$

$$= 6x - x^2 + \frac{x^3}{9} - \frac{x^4}{4(27)} + \cdots + \frac{6(-1)^n x^{n+1}}{(n+1)!3^n} + \cdots$$

$3 : \begin{cases} 1 : \text{two terms} \\ 1 : \text{remaining terms} \\ 1 : \text{general term} \end{cases}$

$\langle -1 \rangle$ missing factor of 6

(c) $f'(x) = -2e^{-x/3}$, so $h(x) = -2k\,e^{-ax/3}$

$$h(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots = e^x$$

$$-2k\,e^{-ax/3} = e^x$$

$$\frac{-a}{3} = 1 \text{ and } -2k = 1$$

$$a = -3 \text{ and } k = -\frac{1}{2}$$

OR

$$f'(x) = -2 + \frac{2}{3}x + \cdots, \text{ so}$$

$$h(x) = kf'(ax) = -2k + \frac{2}{3}akx + \cdots$$

$$h(x) = 1 + x + \cdots$$

$$-2k = 1 \text{ and } \frac{2}{3}ak = 1$$

$$k = -\frac{1}{2} \text{ and } a = -3$$

$3 : \begin{cases} 1 : \text{computes } k f'(ax) \\ 1 : \text{recognizes } h(x) = e^x, \\ \quad\quad \text{or} \\ \quad \text{equates 2 series for } h(x) \\ 1 : \text{values for } a \text{ and } k \end{cases}$

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Work for problem 6(a)

$f(x) = 6e^{-\frac{x}{3}}$

$f'(x) = 6\cdot\left(-\frac{1}{3}\right) e^{-\frac{x}{3}}$

$f''(x) = 6\cdot\left(-\frac{1}{3}\right)^2 e^{-\frac{x}{3}}$

$\vdots$

$f^{(n)}(x) = 6\cdot(-1)^n\cdot\left(\frac{1}{3}\right)^n e^{-\frac{x}{3}}$

$= \boxed{6 - 2x + \frac{1}{3}x^2 - \frac{1}{27}x^3}$

$P(x) = \boxed{6 + 6\left(-\frac{1}{3}\right)x + \frac{6\cdot\left(-\frac{1}{3}\right)^2}{2!}x^2 + \frac{6\cdot\left(-\frac{1}{3}\right)^3}{3!}x^3}$ ← the first four nonzero terms

$G(x) = \boxed{\sum_{n=0}^{\infty} \frac{6\cdot(-1)^n}{n!\cdot 3^n}x^n}$ ← the general term for the Taylor series for f about $x=0$.

Work for problem 6(b)

$g(x) = \int_0^x f(t)\,dt$

$= \int_0^x \sum_{n=0}^{\infty} \frac{6(-1)^n}{n!\cdot 3^n} t^n\,dt$

$= \Big|_0^x\ 6t + 6\left(-\frac{1}{3}\right)\frac{t^2}{2} + \frac{6\cdot\left(-\frac{1}{3}\right)^2}{2!}\frac{t^3}{3} + \frac{6\cdot\left(-\frac{1}{3}\right)^3}{3!}\frac{t^4}{4} + \cdots$

$= \boxed{6x - x^2 + \frac{1}{9}x^3 - \frac{1}{4\times 27}x^4}$ ← the first four nonzero terms

$= \boxed{\sum_{n=0}^{\infty} \frac{6(-1)^n}{n!\cdot 3^n}\,\frac{x^{n+1}}{n+1}}$ ← the general term for the Taylor series for f about $x=0$.

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Work for problem 6(c)

~~$h(x) = e^{x}$~~

We can know that $h(x) = e^{x}$.

$f'(x) = 6 \cdot \left(-\frac{1}{3}\right) e^{-\frac{x}{3}}$

$\therefore f'(ax) = -2e^{-\frac{a}{3}x}$

$\therefore h(x) = k f'(ax)$

$\Rightarrow e^{x} = k \cdot (-2) e^{-\frac{a}{3}x}$

$\Rightarrow e^{(1+\frac{a}{3})x} = -2k$

Since a, k should be independent from x,

$1 + \frac{a}{3} = 0, \quad 1 = -2k$

$\therefore \boxed{a = -3, \; k = -\frac{1}{2}}$

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NO CALCULATOR ALLOWED

AC6
6B1

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Work for problem 6(a)

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(x)}{n!}(x-a)^n$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$6e^{-\frac{x}{3}} = 6\sum_{n=0}^{\infty} \frac{(-\frac{x}{3})^n}{n!} = \boxed{6\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{3^n n!}}$$

27×6

ii) First four

$$\boxed{6, -2x, \frac{x^2}{3}, -\frac{x^3}{27}}$$

Work for problem 6(b)

$$g(x) = \int_0^x f(t)dt$$

$u = -\frac{t}{3}$
$du = -\frac{1}{3}dt$

$$g(x) = 6\int_0^x e^{-\frac{t}{3}} dt$$

$$= -18\int e^u du = \left[-18e^{(-\frac{t}{3})}\right]_0^x = \boxed{-18e^{(-\frac{x}{3})} + 18}$$

$$-18e^{(-\frac{x}{3})} = -18\sum_{n=0}^{\infty} \frac{(-\frac{x}{3})^n}{n!} = \boxed{-18\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{3^n n!} + 18}$$

ii) First 4 Terms

$$0, \frac{+\cancel{18}^{6} \cdot -1x}{\cancel{3}} = \boxed{6x+18}, \frac{-\cancel{18} \cdot x^2}{9 \cdot 2} = \boxed{-x^2+18}, \frac{+\cancel{18}^{2} \cdot -1\,x^3}{9\;27 \cdot \cancel{6}} = \boxed{\frac{x^3}{9}+18}, \frac{-\cancel{18}^{9}\,x^4}{981 \cdot 24} = \boxed{-\frac{x^4}{18}+18}$$

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BC6
6B2

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Work for problem 6(c)

i) $h(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$

ii) $f'(x) = 6 \cdot -\frac{1}{3} e^{\frac{-x}{3}} = -2e^{\frac{-x}{3}}$

$f'(ax) = -2e^{\frac{-ax}{3}}$

$e^x = -2k\, e^{\frac{-ax}{3}}$

$\boxed{\therefore k = -\frac{1}{2}, \quad a = -3}$

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BC6 6C1

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Work for problem 6(a)

$f(x) = 6e^{-x/3} = 6e^{-\frac{x}{3}}$

$T(0) = f(0) + \frac{f'(0)x}{1!} + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \cdots \frac{f^n(0)}{n!}$

$= 6 - 2 + \frac{2}{3} - \frac{2}{9} + \cdots + (-1)^n \frac{2}{3^{n-1}} + \cdots$

a) $f(0) = 6e^0 = 6$

$f'(x) = 6e^{-\frac{x}{3}} \cdot -\frac{1}{3}$

$f'(0) = -2$

$T(0) = 6 - \frac{2x}{1!} + \frac{2x^2}{3\,2!} - \frac{2x^3}{9\,3!} + \cdots + \frac{(-1)^n 2x^n}{3^{n-1} n!} + \cdots$

$f''(x) = -2e^{-\frac{x}{3}} \cdot -\frac{1}{3}$

$f''(0) = \frac{2}{3}$

$f'''(x) = \frac{2}{3} e^{-\frac{x}{3}} \cdot -\frac{1}{3}$

$f'''(0) = -\frac{2}{9}$

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Work for problem 6(b)

$g(x) = \int_0^x f(t)\,dt$

b) $g(x) = \int_0^x 6 + \underline{2x} + \frac{x^2}{3} - \frac{x^3}{27} + \cdots + \frac{(-1)^n 2x^n}{3^{n-1} n!} + \cdots dx$

$g(x) = 6x + x^2 + \frac{x^3}{9} - \frac{x^4}{108} + \cdots + \frac{(-1)^n 2n x^{n-1}}{3^{n-1} n!} + \cdots dx$

$\frac{2}{54} \times x = 4$

$2x = 216$

$x = 108$

$4)\overline{108}$ 27

$2)\overline{54}$ 27

$\frac{1}{27} \times x = 4$

$x =$

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BC6
6C 2

Work for problem 6(c)

$h(x) = kf'(ax)$

$h(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \frac{x^n}{n!} + \cdots$

$f'(x) = -\frac{1}{3}6e^{-\frac{x}{3}}$

$f''(x) = \frac{1}{9}6e^{-\frac{x}{3}}$

$= \frac{2}{3}e^{-\frac{x}{3}}$

$1 = kf'(ax)$

$1 = k(-\frac{1}{3}6e^{-\frac{x}{3}})$

$- \quad 1 = -k2e^{-\frac{x}{3}}$

$-1 = k2e^{-\frac{x}{3}}$

$\frac{-1}{2e^{-x/3}} = k$

$k = -\frac{1}{2}e^{x/3}$

$h'(x) = kf''(ax) \cdot a$

$x = k\frac{2}{3}e^{-ax/3} \cdot a$

$x = (-\frac{e^{x/3}}{\cancel{x}})\frac{x}{3}\frac{a}{e^{\frac{ax}{3}}}$

$-3x = \frac{\cancel{-}e^{x/3}a}{e^{\frac{ax}{3}}}$

$-3x = ae^{x/3 - \frac{ax}{3}}$

$-3x = ae^{\frac{x(1-a)}{3}}$

$-\ln 3x = a\frac{x(1-a)}{3}$

$-3\ln 3x = ax(1-a)$

$= ax - a^2x$

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Question 6

Sample: 6A
Score: 9

The student earned all 9 points.

Sample: 6B
Score: 6

The student earned 6 points: 3 points in part (a), no points in part (b), and 3 points in part (c). Correct work is presented in parts (a) and (c). In part (b) the student adds 18 to each of the first four nonzero terms and thus did not earn the first 2 points. The student does not correctly integrate the general term from part (a), and so the third point was not earned.

Sample: 6C
Score: 4

The student earned 4 points: 3 points in part (a), 1 point in part (b), and no points in part (c). Correct work is presented in part (a). In part (b) the first, third, and fourth terms are correct, but the student makes an error on the sign of the second term. The student earned 1 of the first 2 points. The general term from part (a) is not correctly integrated, and so the third point was not earned. In part (c) the student does not compute $k f'(ax)$ correctly and does not recognize $h(x)$ as the series for e^x.

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