AP® CALCULUS BC
2009 SCORING GUIDELINES (Form B)

Question 6

The function f is defined by the power series

$$f(x) = 1 + (x+1) + (x+1)^2 + \cdots + (x+1)^n + \cdots = \sum_{n=0}^{\infty} (x+1)^n$$

for all real numbers x for which the series converges.

(a) Find the interval of convergence of the power series for f. Justify your answer.

(b) The power series above is the Taylor series for f about $x = -1$. Find the sum of the series for f.

(c) Let g be the function defined by $g(x) = \int_{-1}^{x} f(t)\,dt$. Find the value of $g\left(-\frac{1}{2}\right)$, if it exists, or explain why $g\left(-\frac{1}{2}\right)$ cannot be determined.

(d) Let h be the function defined by $h(x) = f\left(x^2 - 1\right)$. Find the first three nonzero terms and the general term of the Taylor series for h about $x = 0$, and find the value of $h\left(\frac{1}{2}\right)$.

(a) The power series is geometric with ratio $(x+1)$. The series converges if and only if $\lvert x+1 \rvert < 1$. Therefore, the interval of convergence is $-2 < x < 0$.	3 : { 1 : identifies as geometric; 1 : $\lvert x+1 \rvert < 1$; 1 : interval of convergence }
OR	OR
$\lim\limits_{n\to\infty} \left\lvert \frac{(x+1)^{n+1}}{(x+1)^n} \right\rvert = \lvert x+1 \rvert < 1$ when $-2 < x < 0$ At $x = -2$, the series is $\sum_{n=0}^{\infty} (-1)^n$, which diverges since the terms do not converge to 0. At $x = 0$, the series is $\sum_{n=0}^{\infty} 1$, which similarly diverges. Therefore, the interval of convergence is $-2 < x < 0$.	3 : { 1 : sets up limit of ratio; 1 : radius of convergence; 1 : interval of convergence }
(b) Since the series is geometric, $f(x) = \sum_{n=0}^{\infty} (x+1)^n = \frac{1}{1-(x+1)} = -\frac{1}{x}$ for $-2 < x < 0$.	1 : answer
(c) $g\left(-\frac{1}{2}\right) = \int_{-1}^{-\frac{1}{2}} -\frac{1}{x}\,dx = -\ln\lvert x \rvert \Big\rvert_{x=-1}^{x=-\frac{1}{2}} = \ln 2$	2 : { 1 : antiderivative; 1 : value }
(d) $h(x) = f\left(x^2 - 1\right) = 1 + x^2 + x^4 + \cdots + x^{2n} + \cdots$ $h\left(\frac{1}{2}\right) = f\left(-\frac{3}{4}\right) = \frac{4}{3}$	3 : { 1 : first three terms; 1 : general term; 1 : value of $h\left(\frac{1}{2}\right)$ }

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Work for problem 6(a)

By the Ratio Test, $\lim_{n\to\infty}\left|\frac{(x+1)^{n+1}}{(x+1)^n}\right| = |x+1| < 1$.

or $-2 < x < 0$. The radius of convergence is 1.

Now consider both endpoints.

When $x=-2$, $f(x) = 1-1+1-1+\cdots$, which diverges;

when $x=0$, $f(x) = 1+1+1+\cdots$, which also diverges.

Thus, the interval of convergence of f is

$(-2, 0)$.

Work for problem 6(b)

$$f(x) = 1+(x+1)+(x+1)^2+\cdots+(x+1)^n+\cdots$$

$$= \frac{1}{1-(x+1)}$$

$$= -\frac{1}{x}$$

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Work for problem 6(c)

$$g(-\tfrac{1}{2}) = \int_{-1}^{-\frac{1}{2}} f(t)\,dt = -\int_{-1}^{-\frac{1}{2}} \frac{dt}{t} = -\ln|t|\Big|_{-1}^{-\frac{1}{2}} = \ln 2$$

and $-\frac{1}{2}$ is within the interval of convergence.

Work for problem 6(d)

$$h(x) = f(x^2-1) = 1 + x^2 + x^4 + \cdots + x^{2n} + \cdots$$

$$h(\tfrac{1}{2}) = \frac{1}{1-\frac{1}{4}} = \frac{4}{3}$$

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Work for problem 6(a)

$-1 < x+1 < 1$ (in order to converge $\sum_{n=1}^{\infty}(x+1)^n$, $x+1$ has to be between -1 and 1.)

$\therefore \underline{-2 < x < 0}$

Work for problem 6(b)

$f(x) = 1 + (x+1) + (x+1)^2 + \cdots + (x+1)^n$

$= 1 \times \frac{1}{1-(x+1)} = \frac{1}{-x}$

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Work for problem 6(c)

$\int_{-1}^{x} f(t)\,dt = g(x)$

$= x + \frac{(x+1)^2}{2} + \frac{(x+1)^3}{3} + \cdots$

$g(-\frac{1}{2}) = -\frac{1}{2} + \frac{(\frac{1}{2})^2}{2} + \frac{(\frac{1}{2})^3}{3} + \cdots$

it can't be determined, because the ~~general term~~ $g(x)$ is $\sum_{n=1}^{\infty} (\frac{1}{2})^{n+1} \frac{1}{(n+1)}$ which does not converge.

Work for problem 6(d)

$h(x) = f(x^2 - 1)$

$= 1 + (x^2) + (x^2)^2 + \cdots + (x^2)^n$

$h(\frac{1}{2}) = 1 + (\frac{1}{2})^2 + (\frac{1}{2})^4 + (\frac{1}{2})^6 + \cdots$

$= 1 \times \frac{1}{1 - \frac{1}{4}} = 1 \times \frac{4}{3} = \frac{4}{3}$

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Work for problem 6(a)

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(x+1)^n \cdot (x+1)}{(x+1)^n}\right| = |x+1| < 1$$

$-1 < x+1 < 1$

$-2 < x < 0$

At $x = 0$,

$f(x) = \sum_{n=0}^{\infty} 1^n \rightarrow$ diverges.

At $x = -2$,

$f(x) = \sum_{n=0}^{\infty} (-1)^n \rightarrow$ diverges.

$\therefore$ the interval of convergence is $-2 < x < 0$

Work for problem 6(b)

The series is a geometric sequence.

Because the series converges,

$(x+1) < 1$.

Therefore, the sum of the series is $\frac{a}{1-r} = \frac{1}{1-(x+1)} = \frac{1}{1-x-1}$

$= \frac{1}{-x}$

At $x = -1$,

the sum is 1.

$\therefore 1$.

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Work for problem 6(c)

$g(-\frac{1}{2})$ does not exists.

Because $g(x) = \ln(\frac{1}{2}+x)$.

When $x=-\frac{1}{2}$, $g(x) = \ln(0)$, so the value doesn't exist.

Work for problem 6(d)

$P_3(x) = h(x) + h'(x) \times x + \frac{h''(x) \times x^2}{2!}$

$= f(x^2-1) + f'(x^2-1) \times 2x^2 + \frac{4f''(x^2-1)(x^2+1)x^2}{2}$

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Question 6

Sample: 6A
Score: 9

The student earned all 9 points. Note that in part (b) it was not necessary for students to explain their reasoning beyond using the formula for the sum of a convergent geometric series.

Sample: 6B
Score: 6

The student earned 6 points: 2 points in part (a), 1 point in part (b), no points in part (c), and 3 points in part (d). In part (a) the student did not earn the first point since the series is not identified as geometric. In part (b) the student's work is correct and was sufficient to earn the point. In part (c) the student did not earn any points. The student attempts to work with the series for $f(t)$ instead of the closed form expression $-\frac{1}{t}$. (The student would have been eligible for the first point using this method if the displayed antiderivative terms were all correct and included a correct general term.) In part (d) the student's work is correct.

Sample: 6C
Score: 3

The student earned 3 points: 2 points in part (a), 1 point in part (b), no points in part (c), and no points in part (d). In part (a) the student earned the second and third points, using the first method in the scoring guidelines. The series is not identified as geometric. In part (b) the student's work is correct and was sufficient to earn the point. The additional statement concerning the sum at $x = -1$ was ignored.