## Question 5

| $x$ | 2 | 3 | 5 | 8 | 13 |
|---|---|---|---|---|---|
| $f(x)$ | 1 | 4 | –2 | 3 | 6 |

Let $f$ be a function that is twice differentiable for all real numbers. The table above gives values of $f$ for selected points in the closed interval $2 \le x \le 13$.

(a) Estimate $f'(4)$. Show the work that leads to your answer.

(b) Evaluate $\int_2^{13} (3 - 5f'(x))\,dx$. Show the work that leads to your answer.

(c) Use a left Riemann sum with subintervals indicated by the data in the table to approximate $\int_2^{13} f(x)\,dx$. Show the work that leads to your answer.

(d) Suppose $f'(5) = 3$ and $f''(x) < 0$ for all $x$ in the closed interval $5 \le x \le 8$. Use the line tangent to the graph of $f$ at $x = 5$ to show that $f(7) \le 4$. Use the secant line for the graph of $f$ on $5 \le x \le 8$ to show that $f(7) \ge \frac{4}{3}$.

---

(a) $f'(4) \approx \dfrac{f(5) - f(3)}{5 - 3} = -3$

1 : answer

(b) $\int_2^{13} (3 - 5f'(x))\,dx = \int_2^{13} 3\,dx - 5\int_2^{13} f'(x)\,dx$
$= 3(13 - 2) - 5(f(13) - f(2)) = 8$

2 : $\begin{cases} 1 : \text{uses Fundamental Theorem of Calculus} \\ 1 : \text{answer} \end{cases}$

(c) $\int_2^{13} f(x)\,dx \approx f(2)(3-2) + f(3)(5-3) + f(5)(8-5) + f(8)(13-8) = 18$

2 : $\begin{cases} 1 : \text{left Riemann sum} \\ 1 : \text{answer} \end{cases}$

(d) An equation for the tangent line is $y = -2 + 3(x - 5)$. Since $f''(x) < 0$ for all $x$ in the interval $5 \le x \le 8$, the line tangent to the graph of $y = f(x)$ at $x = 5$ lies above the graph for all $x$ in the interval $5 < x \le 8$.

Therefore, $f(7) \le -2 + 3 \cdot 2 = 4$.

An equation for the secant line is $y = -2 + \frac{5}{3}(x - 5)$. Since $f''(x) < 0$ for all $x$ in the interval $5 \le x \le 8$, the secant line connecting $(5, f(5))$ and $(8, f(8))$ lies below the graph of $y = f(x)$ for all $x$ in the interval $5 < x < 8$.

Therefore, $f(7) \ge -2 + \frac{5}{3} \cdot 2 = \frac{4}{3}$.

4 : $\begin{cases} 1 : \text{tangent line} \\ 1 : \text{shows } f(7) \le 4 \\ 1 : \text{secant line} \\ 1 : \text{shows } f(7) \ge \frac{4}{3} \end{cases}$

**NO CALCULATOR ALLOWED**

| $x$ | 2 | 3 | 5 | 8 | 13 |
|---|---|---|---|---|---|
| $f(x)$ | 1 | 4 | –2 | 3 | 6 |

5A1

Work for problem 5(a)

$$f'(4) \approx \frac{-2-4}{5-3} = \frac{-6}{2} = -3$$

Work for problem 5(b)

$$\int_2^{13} 3-5f'(x)\,dx = \int_2^{13} 3\,dx - 5\int_2^{13} f'(x)\,dx$$

$$3x\Big|_2^{13} - 5\left(f(13)-f(2)\right)$$

$$33 - 5(6-1)$$

$$33 - 5(5)$$

$$33 - 25 = 8$$

Do not write beyond this border.

**NO CALCULATOR ALLOWED**

5A2

Work for problem 5(c)

$(3-2)\cdot 1 + (5-3)\cdot 4 + (8-5)\cdot -2 + (13-8)\cdot 3$

$1+8-6+15=18$

Work for problem 5(d)

$y+2=3(x-5)$

$y+2=3(7-5)$

$y+2=3(2)$

$y=4$

since $f''(x)<0$ the tangent line is an overapproximation so $f(7)\leq 4$

$\frac{3--2}{8-5}=\frac{5}{3}$

$y+2=\frac{5}{3}(x-5)$

$y+2=\frac{5}{3}(7-5)$

$y+2=\frac{5}{3}(2)$

$y+2=\frac{10}{3}$

$y=\frac{10}{3}-2$

$y=\frac{4}{3}$

since $f''(x)<0$ the secant line is an underapproximation at $f(7)$, so $f(7)$ must be $\geq\frac{4}{3}$

Do not write beyond this border.

5 5 5 5 5 5 5 5 5 5

**NO CALCULATOR ALLOWED**

5B1

| $x$ | 2 | 3 | 5 | 8 | 13 |
|---|---|---|---|---|---|
| $f(x)$ | 1 | 4 | –2 | 3 | 6 |

Do not write beyond this border.

Work for problem 5(a)

$$f'(4) = \frac{f(5)-f(3)}{5-3}$$

$$f'(4) = \frac{-2-4}{5-3}$$

$$f'(4) = \frac{-6}{2} = -3$$

Work for problem 5(b)

$$\int_2^{13} (3 - 5f'(x))\,dx$$

$$\int_2^{13} 3\,dx - 5\int_2^{13} f'(x)\,dx$$

$$3x\Big|_2^{13} - 5\left(f(x)\Big|_2^{13}\right)$$

$$3(13) - 3(2) - 5(f(13) - f(2))$$

$$26 - 6 - 5(6-1)$$

$$20 - 5(5)$$

$$20 - 25$$

$$\int_2^{13} (3 - 5f'(x))\,dx = 5$$

**NO CALCULATOR ALLOWED**

5B2

Work for problem 5(c)

$\int_2^{13} f(x)\,dx$ using left Riemann sums:

$(1\cdot 2)+(2\cdot 4)+(3\cdot -2)+(5\cdot 3)$

$2+8-6+15=19$

$\int_2^{13} f(x)\,dx \approx 19$

Work for problem 5(d)

$f'(5)=3$ and $f(5)=-2$

$y+2=3(x-5)$

$y=3x-15-2$

$y=3x-17$

at $x=7$: $y=3(7)-17$

$y=21-17$

$y=4$

Because $f''(x)<0$, $f'(x)$ is decreasing over the interval $5\le x\le 8$. This means $f'(5)$ is the largest value over this interval, so $f(7)$ can not be any greater than 4. Therefore, $f(7)\le 4$.

$f'(c)=\frac{f(8)-f(5)}{8-5}$

$f'(c)=\frac{3--2}{3}=\frac{5}{3}$

$y+2=\frac{5}{3}(x-5)$

at $x=7$: $y+2=\frac{5}{3}(7-5)$

$y+2=\frac{10}{3}$

$y=\frac{4}{3}$

Because this secant line is the average slope of the interval $5\le x\le 8$, it is an underapproximation of $f(7)$. Therefore $f(7)\ge \frac{4}{3}$.

Do not write beyond this border.

**NO CALCULATOR ALLOWED**

5C1

| $x$ | 2 | 3 | 5 | 8 | 13 |
|---|---|---|---|---|---|
| $f(x)$ | 1 | 4 | −2 | 3 | 6 |

Work for problem 5(a)

$f'(4) = \frac{-2-1}{5-2} = \frac{-3}{3} = -1$

$\boxed{f'(4) = -1}$

Work for problem 5(b)

$\int_2^{13} (3 - 5f'(x))\,dx$

$\Big|_2^{13} 3x - 5(f(x))$

$13(3) - 6$

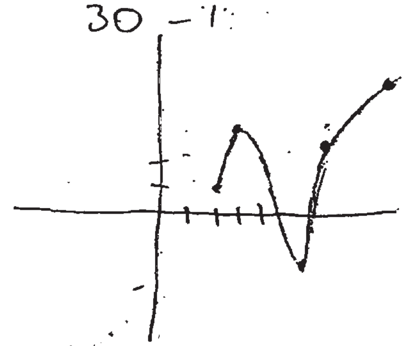
$39 - 6 = 33$

$33 - 29 = 4$

$\Big|_2^{13} 5f(x)$

$5(f(13)) - 5(f(2))$

$30 - 1$

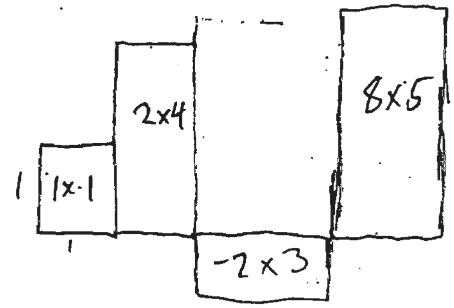


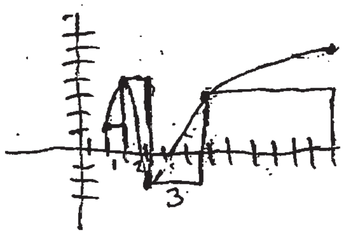
Do not write beyond this border.

**NO CALCULATOR ALLOWED**

5C8

**Work for problem 5(c)**



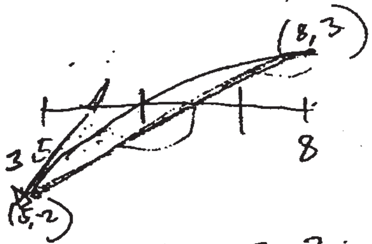




$1+8-6+40$

$9-6$ $43$

**Work for problem 5(d)**

$f'(5)=3$




$y+2=3(x-5)$

$y+2=3x-15$

$y=3x-17$

$y=3(7)-17$

$y=21-17$

$y=4$ $\leq 4$

$f(7)$

$\frac{3+2}{8-5}=\frac{5}{3}$

$y+2=\frac{5}{3}(x-5)$

$y+2=\frac{5x}{3}-\frac{25}{3}$

$y=\frac{5x}{3}-\frac{31}{3}$

$f(7)\geq\frac{4}{3}$

$y=\frac{35}{3}-\frac{31}{3}=\frac{4}{3}\geq\frac{4}{3}$



Do not write beyond this border.

## Question 5

### Overview

This problem presented students with a table of values for a function $f$ sampled at five values of $x$. It was also stated that $f$ is twice differentiable for all real numbers. Part (a) asked for an estimate for $f'(4)$. Since $x = 4$ falls between the values sampled on the table, students should have calculated the slope of the secant line to the graph of $f$ corresponding to the closest pair of points in the supplied data that brackets $x = 4$. Part (b) tested students' ability to apply properties of the definite integral to evaluate $\int_2^{13} (3 - 5f'(x))\, dx$. Part (c) asked for an approximation to $\int_2^{13} f(x)\, dx$ using the subintervals of $[2, 13]$ indicated by the data in the table. In part (d) it was also stated that $f'(5) = 3$ and $f''(x) < 0$ for all $x$ in $[5, 8]$. Students were asked to use the line tangent to the graph of $f$ at $x = 5$ to show that $f(7) \le 4$ and to use the secant line for the graph of $f$ on $5 \le x \le 8$ to show that $f(7) \ge \frac{4}{3}$. For the former inequality, students should have used the fact that $f''$ is negative (so $f'$ is decreasing) on $[5, 8]$ so that the tangent line at $x = 5$ lies above the graph of $f$ throughout $(5, 8]$. For the latter inequality, students should have used the sign of $f''$ to conclude that the indicated secant line lies below the graph of $f$ for $5 < x < 8$; in particular, the point on the graph of the secant line corresponding to $x = 7$ is below the corresponding point on the graph of $f$.

**Sample: 5A**
**Score: 9**

The student earned all 9 points. In part (b) the student's second line earned the first point, and the third line earned the second point. In part (c) the student's first line earned both points. In part (d) the student's first line earned the first point. The second point was earned by showing that $y = 4$ when $x = 7$ on the tangent line, stating the desired inequality $f(7) \le 4$, and giving an acceptable reason to validate the inequality. The third and fourth points were earned in a similar manner using the secant line.

**Sample: 5B**
**Score: 6**

The student earned 6 points: 1 point in part (a), 1 point in part (b), 1 point in part (c), and 3 points in part (d). In part (a) the student's second line earned the point. In part (b) the student's fourth line earned the first point for use of the Fundamental Theorem of Calculus. The student makes subsequent errors. In part (c) the student's second line earned the first point since seven of the eight presented factors are correct. The student did not earn the answer point. In part (d) the student's second line on the left earned the first point. The second point was earned by showing that $y = 4$ when $x = 7$ on the tangent line, stating the desired inequality $f(7) \le 4$, and giving an acceptable reason to validate the inequality. The student's third line on the right earned the third point. The last point was not earned since the student's reason does not validate the inequality $f(7) \ge \frac{4}{3}$.

## Question 5 (continued)

**Sample: 5C**
**Score: 4**

The student earned 4 points: no points in part (a), 1 point in part (b), 1 point in part (c), and 2 points in part (d). In part (a) the student's answer is incorrect. In part (b) the student earned the first point by correctly applying the Fundamental Theorem of Calculus to the derivative of $f$. The student makes a subsequent arithmetic error. In part (c) the student earned the first point since seven of the eight presented factors are correct. The student did not earn the answer point. In part (d) the student earned the first and third points for correct equations for the tangent and secant lines. Since the student does not explain why either of the two inequalities is valid, the student did not earn the other points.