Question 6

The Maclaurin series for the function f is given by $f(x) = \sum_{n=2}^{\infty} \frac{(-1)^n (2x)^n}{n-1}$ on its interval of convergence.

(a) Find the interval of convergence for the Maclaurin series of f. Justify your answer.

(b) Show that $y = f(x)$ is a solution to the differential equation $xy' - y = \frac{4x^2}{1+2x}$ for $|x| < R$, where R is the radius of convergence from part (a).

(a) $$\lim_{n\to\infty} \left| \frac{\dfrac{(2x)^{n+1}}{(n+1)-1}}{\dfrac{(2x)^n}{n-1}} \right| = \lim_{n\to\infty} \left| 2x \cdot \frac{n-1}{n} \right| = \lim_{n\to\infty} \left| 2x \cdot \frac{n-1}{n} \right| = |2x|$$

$|2x| < 1$ for $|x| < \frac{1}{2}$

Therefore the radius of convergence is $\frac{1}{2}$.

When $x = -\frac{1}{2}$, the series is $\sum_{n=2}^{\infty} \frac{(-1)^n (-1)^n}{n-1} = \sum_{n=2}^{\infty} \frac{1}{n-1}$.
This is the harmonic series, which diverges.

When $x = \frac{1}{2}$, the series is $\sum_{n=2}^{\infty} \frac{(-1)^n 1^n}{n-1} = \sum_{n=2}^{\infty} \frac{(-1)^n}{n-1}$.
This is the alternating harmonic series, which converges.

The interval of convergence for the Maclaurin series of f is $\left(-\frac{1}{2}, \frac{1}{2}\right]$.

$$5 : \begin{cases} 1 : \text{sets up ratio} \\ 1 : \text{limit evaluation} \\ 1 : \text{radius of convergence} \\ 1 : \text{considers both endpoints} \\ 1 : \text{analysis and interval of convergence} \end{cases}$$

(b) $$y = \frac{(2x)^2}{1} - \frac{(2x)^3}{2} + \frac{(2x)^4}{3} - \cdots + \frac{(-1)^n (2x)^n}{n-1} + \cdots$$
$$= 4x^2 - 4x^3 + \frac{16}{3}x^4 - \cdots + \frac{(-1)^n (2x)^n}{n-1} + \cdots$$
$$y' = 8x - 12x^2 + \frac{64}{3}x^3 - \cdots + \frac{(-1)^n n(2x)^{n-1} \cdot 2}{n-1} + \cdots$$
$$xy' = 8x^2 - 12x^3 + \frac{64}{3}x^4 - \cdots + \frac{(-1)^n n(2x)^n}{n-1} + \cdots$$
$$xy' - y = 4x^2 - 8x^3 + 16x^4 - \cdots + (-1)^n (2x)^n + \cdots$$
$$= 4x^2 \left(1 - 2x + 4x^2 - \cdots + (-1)^n (2x)^{n-2} + \cdots\right)$$

The series $1 - 2x + 4x^2 - \cdots + (-1)^n (2x)^{n-2} + \cdots = \sum_{n=0}^{\infty} (-2x)^n$ is a geometric series that converges to $\frac{1}{1+2x}$ for $|x| < \frac{1}{2}$. Therefore $xy' - y = 4x^2 \cdot \frac{1}{1+2x}$ for $|x| < \frac{1}{2}$.

$$4 : \begin{cases} 1 : \text{series for } y' \\ 1 : \text{series for } xy' \\ 1 : \text{series for } xy' - y \\ 1 : \text{analysis with geometric series} \end{cases}$$

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Work for problem 6(a)

$f(x) = \sum_{n=2}^{\infty} \frac{(-1)^n (2x)^n}{n-1}$

Interval of convergence?

Let's use the Ratio Test

$\lim_{n\to\infty} \left| \frac{\frac{(-1)^{n+1}(2x)^{n+1}}{n}}{\frac{(-1)^n (2x)^n}{n-1}} \right| = \lim_{n\to\infty} |(-1)(2x)| = |2x| < 1$

$|x| < \frac{1}{2}$ or $-\frac{1}{2} < x < \frac{1}{2}$ → convergent

$x = \frac{1}{2}$ → $f(x) = \sum_{n=2}^{\infty} \frac{(-1)^n}{n-1}$

By Leibniz's criteria on convergence on series of alternative terms. $a_n > a_{n+1}$. $f(x)$ converges.

$x = -\frac{1}{2}$ $f(x) = \sum_{n=2}^{\infty} \frac{1}{n-1}$ → divergent.

$-\frac{1}{2} < x \leq \frac{1}{2}$ $(-\frac{1}{2}, \frac{1}{2}]$ is the interval of convergence for the Maclaurin series of f.

$R = \frac{1}{2}$

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Work for problem 6(b)

$$xy'-y=\frac{4x^2}{1+2x}$$

$$f(x)=\sum_{n=2}^{\infty}\frac{(-1)^n(2x)^n}{n-1}$$

$$f'(x)=\sum_{n=2}^{\infty}\frac{(-1)^n 2^n}{n-1}\cdot n\cdot x^{n-1}$$

$$xy'-y=\sum_{n=2}^{\infty}\frac{(-1)^n 2^n}{n-1}\cdot n\cdot x^n-\sum_{n=2}^{\infty}\frac{(-1)^n(2x)^n}{n-1}$$

$$=\sum_{n=2}^{\infty}\frac{(-1)^n(2x)^n(n-1)}{n-1}=\sum_{n=2}^{\infty}(-2x)^n$$

since $|x|<\frac{1}{2}$, $\sum_{n=2}^{\infty}(-2x)^n$ Converges (Ratio Test)

$\sum_{n=2}^{\infty}(-2x)^n$ is a geometric series $\left(1+m+\cdots+m^n=\frac{1-m^{n+1}}{1-m}\right)$

$\left(\lim_{n\to\infty}\sum_{k=0}^{n}m^k=\frac{1}{1-m}\right)$

$$\therefore \sum_{n=2}^{\infty}(-2x)^n=\sum_{n=0}^{\infty}(-2x)^n-1-(-2x)=\frac{1}{1+2x}-1+2x$$

$$=\frac{1-1-2x+2x+4x^2}{1+2x}=\frac{4x^2}{1+2x}$$

Therefore $y=f(x)$ is a solution to $xy'-y=\frac{4x^2}{1+2x}$ for $|x|<R$

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Work for problem 6(a)

$$f(x) = \sum_{n=2}^{\infty} \frac{(-1)^n (2x)^n}{n-1}$$

$$\lim_{n\to\infty} \left| \left[\frac{\frac{\cancel{(-1)^{n+1}} \cancel{(2x)^{n+1}}}{n+1-1}}{\frac{\cancel{(-1)^n} \cancel{(2x)^n}}{n-1}} \right] \right|$$

$$= \lim_{n\to\infty} \left| \frac{(-1)(2x)(n-1)}{n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{n-1}{n} \cdot 2x \right|$$

$$= \lim_{n\to\infty} \left| \frac{1-\frac{1}{n}}{1} \cdot 2x \right|$$

$$= |2x|$$

~~For~~ By ratio test,

series is convergent when $|2x| < 1$

$-\frac{1}{2} < x < \frac{1}{2}$

When $x = -\frac{1}{2}$

$f(x) = \sum_{n=2}^{\infty} \frac{(-1)^n (-1)^n}{n-1} = \sum_{n=2}^{\infty} \frac{1}{n-1} = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges (P-series)

When $x = \frac{1}{2}$

$f(x) = \sum_{n=2}^{\infty} \frac{(-1)^n (1)^n}{n-1} = \sum_{n=2}^{\infty} (-1)^n \frac{1}{n-1}$ converges (alternative series)

Hence, the interval of convergence for $f(x)$ is $-\frac{1}{2} < x \le \frac{1}{2}$

Continue problem 6 on page 15.

Work for problem 6(b)

$$y = f(x) = \sum_{n=2}^{\infty} \frac{(-1)^n (2x)^n}{n-1}$$

$$y = (2x)^2 - \frac{(2x)^3}{2} + \frac{(2x)^4}{3} \cdots \frac{(-1)^n (2x)^n}{n-1}$$

$$y' = 2\cdot 2(2x) - \frac{2\cdot 3(2x)^2}{2} + \frac{2\cdot 4(2x)^3}{3} \cdots \frac{(-1)^n \cdot 2n \cdot (2x)^{n-1}}{n-1}$$

$$x y' = \frac{(-1)^n \cdot (2n)(2x)}{n-1}$$

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Work for problem 6(a)

$$\lim_{n\to\infty} \frac{(n+1)\text{th term}}{n\text{th term}} = \lim_{n\to\infty} \frac{\frac{(-1)^{n+1}(2x)^{n+1}}{n}}{\frac{(-1)^n (2x)^n}{n-1}}$$

$$= \lim_{n\to\infty} \frac{-2x(n-1)}{n} = 0$$

∴ The series must converge

The interval of convergence is all real number

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 6C₂

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Work for problem 6(b)

Putting $y = \sum_{n=2}^{\infty} \frac{(-1)^n (2x)^n}{n-1}$

$$y' = \sum_{n=2}^{\infty} \frac{2n(-1)^n (2x)^{n-1}}{n-1}$$

$$xy' = \sum_{n=2}^{\infty} \frac{n(-1)^n (2x)^n}{n-1}$$

$xy' - y$

$$= x\sum_{n=2}^{\infty} \frac{2n(-1)^n (2x)^{n-1}}{n-1} - \sum_{n=2}^{\infty} \frac{(-1)^n (2x)^n}{n-1}$$

$$= \sum_{n=2}^{\infty} \frac{(-1)^n (2x)^n}{n-1} [n-1]$$

$$= \sum_{n=2}^{\infty} (-1)^n (2x)^n$$

$$= \frac{(-1)^2 (2x)^2}{1+2x}$$

$$= \frac{4x^2}{1+2x}$$

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2010 SCORING COMMENTARY (Form B)

Question 6

Sample: 6A
Score: 9

The student earned all 9 points. In part (a) an ideal solution would include an additional step at the beginning of the limit calculation. The student's presented work is correct.

Sample: 6B
Score: 6

The student earned 6 points: 5 points in part (a) and 1 point in part (b). In part (a) the student's work is correct. In part (b) the student finds the series for y', but what the student presents for xy' is not a series. Only the first point was earned.

Sample: 6C
Score: 4

The student earned 4 points: 1 point in part (a) and 3 points in part (b). In part (a) the student sets up the ratio test but does not evaluate the limit correctly. The first point was earned. In part (b) the student finds the series for y', xy', and $xy' - y$. The first 3 points were earned. The student has an algebraic error in the work leading to $\frac{4x^2}{1 + 2x}$, so the answer point was not earned.