Question 2

The polar curve r is given by $r(\theta) = 3\theta + \sin\theta$, where $0 \le \theta \le 2\pi$.

(a) Find the area in the second quadrant enclosed by the coordinate axes and the graph of r.

(b) For $\frac{\pi}{2} \le \theta \le \pi$, there is one point P on the polar curve r with x-coordinate -3. Find the angle θ that corresponds to point P. Find the y-coordinate of point P. Show the work that leads to your answers.

(c) A particle is traveling along the polar curve r so that its position at time t is $(x(t), y(t))$ and such that $\frac{d\theta}{dt} = 2$. Find $\frac{dy}{dt}$ at the instant that $\theta = \frac{2\pi}{3}$, and interpret the meaning of your answer in the context of the problem.

(a) Area $= \frac{1}{2}\int_{\pi/2}^{\pi} (r(\theta))^2\, d\theta = 47.513$	$3: \begin{cases} 1: \text{integrand} \\ 1: \text{limits and constant} \\ 1: \text{answer} \end{cases}$
(b) $-3 = r(\theta)\cos\theta = (3\theta + \sin\theta)\cos\theta$ $\theta = 2.01692$ $y = r(\theta)\sin(\theta) = 6.272$	$3: \begin{cases} 1: \text{equation} \\ 1: \text{value of } \theta \\ 1: y\text{-coordinate} \end{cases}$
(c) $y = r(\theta)\sin\theta = (3\theta + \sin\theta)\sin\theta$ $\left.\frac{dy}{dt}\right\vert_{\theta=2\pi/3} = \left[\frac{dy}{d\theta}\cdot\frac{d\theta}{dt}\right]_{\theta=2\pi/3} = -2.819$ The y-coordinate of the particle is decreasing at a rate of 2.819.	$3: \begin{cases} 1: \text{uses chain rule} \\ 1: \text{answer} \\ 1: \text{interpretation} \end{cases}$

Work for problem 2(a)

$$\text{Area} = \frac{1}{2}\int_{\frac{\pi}{2}}^{\pi} r^2 \, d\theta$$

$$= \frac{1}{2}\int_{\frac{\pi}{2}}^{\pi} (3\theta + \sin\theta)^2 \, d\theta$$

$$= 47.513$$

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Work for problem 2(b)

$$x = r\cos\theta$$

$$x = (3\theta + \sin\theta)\cos\theta$$

$$-3 = (3\theta + \sin\theta)\cos\theta$$

$$\theta = 2.017 \text{ radians}$$

The y-coordinate of point $P = r\sin\theta$

$$= (3\theta + \sin\theta)\sin\theta$$

$$= 6.272$$

Continue problem 2 on page 7.

Work for problem 2(c)

$$\frac{dy}{dt} = \frac{d(r\sin\theta)}{dt}$$

$$= \frac{d(r\sin\theta)}{d\theta} \cdot \frac{d\theta}{dt}$$

$$= \frac{d}{d\theta}\left[(3\theta + \sin\theta)\cdot\sin\theta\right] \cdot \frac{d\theta}{dt}$$

$$= \frac{d}{d\theta}(3\theta\sin\theta + \sin^2\theta) \cdot \frac{d\theta}{dt}$$

$$= (3\sin\theta + 3\theta\cos\theta + \sin 2\theta) \cdot \frac{d\theta}{dt}$$

$$\left.\frac{dy}{dt}\right|_{\theta=\frac{2\pi}{3}} = \left(3\sin\frac{2\pi}{3} + 3\left(\frac{2\pi}{3}\right)\cos\frac{2\pi}{3} + \sin\frac{4\pi}{3}\right) \cdot 2$$

$$= -2.819$$

$\because$ y is positive at the instant $\theta = \frac{2\pi}{3}$ and $\frac{dy}{d\theta}$ is negative at the instant $\theta = \frac{2\pi}{3}$,

$\therefore$ the particle is travelling towards the x-axis at the instant $\theta = \frac{2\pi}{3}$

END OF PART A OF SECTION II

IF YOU FINISH BEFORE TIME IS CALLED, YOU MAY CHECK YOUR WORK ON PART A ONLY. DO NOT GO ON TO PART B UNTIL YOU ARE TOLD TO DO SO.

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Work for problem 2(a)

when the graph is in the II quadrant, $\theta \in (\frac{\pi}{2}, \pi)$

so area $= \int_{\frac{\pi}{2}}^{\pi} r(\theta)^2 \cdot d\theta = \int_{\frac{\pi}{2}}^{\pi} \frac{1}{2}(3\theta + \sin\theta)^2 \cdot d\theta$

when we include the axes,

$= 49.513$

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Work for problem 2(b)

$r_{(\theta)} = 3\theta + \sin\theta.$

thus $x(\theta) = (3\theta + \sin\theta) \cdot \cos\theta.$

$y(\theta) = (3\theta + \sin\theta) \cdot \sin\theta.$

when $x(\theta) = -3$, $\theta \in [\frac{\pi}{2}, \pi]$, $\theta = 2.017$

$y_{(\theta)} = x(\theta) \cdot \tan\theta = -3 \cdot \tan\theta = 6.271$

so: $\theta = 2.017$

$y(\theta) = 6.271$

Continue problem 2 on page 7.

Work for problem 2(c)

$$\frac{dy}{dt} = \frac{dy}{d\theta} \cdot \frac{d\theta}{dt}$$

$$\frac{dy}{d\theta} = \frac{d(3\theta + \sin\theta) \cdot \sin\theta}{d\theta} = 3\theta \cdot \cos\theta + 3\sin\theta + 2\sin\theta \cdot \cos\theta$$

so when $\theta = \frac{2\pi}{3}$ $\frac{dy}{dt} = 2 \cdot \left[3 \cdot \frac{2\pi}{3} \cdot \left(-\frac{1}{2}\right) + 3 \cdot \frac{\sqrt{3}}{2} + 2 \cdot \frac{\sqrt{3}}{2} \cdot \left(-\frac{1}{2}\right)\right]$

$= 2\sqrt{3} - 2\pi$.

① So when $\theta = \frac{2\pi}{3}$ $\frac{dy}{dt} = 2\sqrt{3} - 2\pi$.

② it means, when $\theta = \frac{2\pi}{3}$, the speed of the particle on the direction $\vec{y}$ is $(2\sqrt{3} - 2\pi)$.

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END OF PART A OF SECTION II

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Work for problem 2(a)

$$A = \int_{\frac{\pi}{2}}^{\pi} \frac{1}{2} r^2 d\theta = \frac{1}{2}\int_{\frac{\pi}{2}}^{\pi} (9\theta^2 + \sin^2\theta + 6\theta\sin\theta)\, d\theta$$

$$= \frac{1}{2}(3\theta^3 +$$

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Work for problem 2(b)

$x = r\cos\theta$

$y = r\sin\theta$

$x = -3, \quad r\cos\theta = -3 \quad , r = 3\theta + \sin\theta$

$3\theta\cos\theta + \sin\theta\cos\theta = -3$

$y = x \tan\theta$

Continue problem 2 on page 7.

Work for problem 2(c)

$$\frac{dy}{dt} = \frac{dy}{d\theta} \times \frac{d\theta}{dt}$$

$$\cancel{= \frac{2}{3}\theta\pi \times 2 = \frac{4}{3}\pi}$$

$$= (\sin\theta + r\cos\theta) \times 2$$

$$= \left(\frac{\sqrt{3}}{2} - \frac{1}{2}r\right) \times 2$$

$$= \sqrt{3} - r$$

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END OF PART A OF SECTION II

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Question 2

Sample: 2A
Score: 9

The student earned all 9 points. Because the particle is above the x-axis, it is sufficient that the student states "the particle is travelling towards the x-axis" in part (c).

Sample: 2B
Score: 6

The student earned 6 points: 2 points in part (a), 2 points in part (b), and 2 points in part (c). In part (a) the student's integral is correct, so the first 2 points were earned. The answer is incorrect. In part (b) the student earned the equation point implicitly and earned the point for the value of $\theta.$ The student's answer is incorrect, possibly as a result of intermediate rounding. In part (c) the student earned the first 2 points. The student does not indicate that the y-coordinate of the particle is decreasing.

Sample: 2C
Score: 4

The student earned 4 points: 2 points in part (a), 1 point in part (b), and 1 point in part (c). In part (a) the student's integral is correct, so the first 2 points were earned. In part (b) the fourth line of the student's solution earned the first point. In part (c) the student earned the chain-rule point.