Question 1

At time t, a particle moving in the xy-plane is at position $(x(t), y(t))$, where $x(t)$ and $y(t)$ are not explicitly given. For $t \geq 0$, $\dfrac{dx}{dt} = 4t + 1$ and $\dfrac{dy}{dt} = \sin\left(t^2\right)$. At time $t = 0$, $x(0) = 0$ and $y(0) = -4$.

(a) Find the speed of the particle at time $t = 3$, and find the acceleration vector of the particle at time $t = 3$.

(b) Find the slope of the line tangent to the path of the particle at time $t = 3$.

(c) Find the position of the particle at time $t = 3$.

(d) Find the total distance traveled by the particle over the time interval $0 \leq t \leq 3$.

(a) Speed $= \sqrt{(x'(3))^2 + (y'(3))^2} = 13.006$ or 13.007 Acceleration $= \langle x''(3), y''(3) \rangle$ $= \langle 4, -5.466 \rangle$ or $\langle 4, -5.467 \rangle$	$2 : \begin{cases} 1 : \text{speed} \\ 1 : \text{acceleration} \end{cases}$
(b) Slope $= \dfrac{y'(3)}{x'(3)} = 0.031$ or 0.032	1 : answer
(c) $x(3) = 0 + \int_0^3 \frac{dx}{dt}\, dt = 21$ $y(3) = -4 + \int_0^3 \frac{dy}{dt}\, dt = -3.226$ At time $t = 3$, the particle is at position $(21, -3.226)$.	$4 : \begin{cases} 2 : x\text{-coordinate} \\ \quad 1 : \text{integral} \\ \quad 1 : \text{answer} \\ 2 : y\text{-coordinate} \\ \quad 1 : \text{integral} \\ \quad 1 : \text{answer} \end{cases}$
(d) Distance $= \int_0^3 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = 21.091$	$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$

CALCULUS BC

SECTION II, Part A

Time—30 minutes

Number of problems—2

A graphing calculator is required for these problems.

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Work for problem 1(a)

$$\text{Speed} = \sqrt{\left(\frac{dy}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2}$$

$$\left.\frac{dy}{dt}\right|_{t=3} = \sin(3^2) = \sin 9$$

$$\left.\frac{dx}{dt}\right|_{t=3} = 4(3)+1 = 13$$

$$\text{Speed}\Big|_{t=3} = \sqrt{(\sin 9)^2 + (13)^2}$$

$$\boxed{\text{Speed} = 13.007}$$

$$\text{Acceleration Vector} = \left\langle \frac{d^2x}{dt^2}, \frac{d^2y}{dt^2} \right\rangle$$

$$\frac{dx}{dt} = 4t+1; \quad \frac{d^2x}{dt^2} = 4$$

$$\frac{dy}{dt} = \sin(t^2); \quad \frac{d^2y}{dt^2} = 2t\cos(t^2)$$

$$\left.\frac{d^2x}{dt^2}\right|_{t=3} = 4 \ ; \quad \left.\frac{d^2y}{dt^2}\right|_{t=3} = -5.467$$

$$\boxed{\text{Acceleration Vector} = \langle 4, -5.467 \rangle}$$

Work for problem 1(b)

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}$$

From part A,

$$\left.\frac{dy}{dt}\right|_{t=3} = \sin 9$$

$$\left.\frac{dx}{dt}\right|_{t=3} = 13$$

$$\frac{dy}{dx} = \frac{\sin 9}{13} = \boxed{0.0317}$$

Continue problem 1 on page

Work for problem 1(c)

x position at $t=3$ is equal to $x(0)+\int_0^3 \frac{dx}{dt}\,dt$

y position at $t=3$ is equal to $y(0)+\int_0^3 \frac{dy}{dt}\,dt$

$x(3)=0+\int_0^3 (4t+1)\,dt = 21$

$y(3)=-4+\int_0^3 \sin(t^2)\,dt = -3.226$

At $t=3$, the particle's position is $\boxed{(21, -3.226)}$

Work for problem 1(d)

Distance Travelled $= \int_0^3 \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$

$= \int_0^3 \sqrt{(4t+1)^2+\sin^2(t^2)}\,dt$

$\boxed{\text{Distance Travelled} = 21.091}$

CALCULUS BC

SECTION II, Part A

Time—30 minutes

Number of problems—2

A graphing calculator is required for these problems.

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Work for problem 1(a)

speed is $|\vec{v}|$ which can be given by the magnitude of the velocity vector $(4t+1, \sin(t^2))$ because its position $\vec{r} = (x(t), y(t))$, velocity $\vec{v} = \left(\frac{dx}{dt}, \frac{dy}{dt}\right)$

at $t=3$: $\vec{v} = (13, 0.412)$, so its speed

$$|\vec{v}| = \sqrt{13^2 + 0.412^2} = \boxed{13.007}$$

Work for problem 1(b)

At $t=3$ the slope of the line tangent to the path of the particle at $t=3$, is given by $\frac{dy}{dx}$ which equals $\frac{dy/dt}{dx/dt}$. At $t=3$:

$$\left.\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right|_{t=3} \Rightarrow \left.\frac{\sin(t^2)}{4t+1}\right|_{t=3} \Rightarrow \frac{\sin 9}{13} = \boxed{0.0317}$$

(slope of tangent line at $t=3$)

Continue problem 1 on page

Work for problem 1(c)

Initial position at $t=0$: $x=0$, $y=-4$

Thus $\vec{r} = \int \vec{v}\,dt \Rightarrow \left(\int (4t+1)\,dt, \int \sin(t^2)\,dt \right)$ which

is $\vec{r} \Rightarrow \left(2t^2+t+C, \frac{-\cos t^2}{2t}+C\right)$...

solving for C, at $t=0$

for x:
$0=2t^2+t+C$
$C=0$

for y:
$-4=-\frac{\cos t^2}{2t}+C$
$-8t=-\cos t^2+C$
$0=-1+C$
$C=1$

$\Rightarrow$ so $\vec{r}=\left(2t^2+t, -\frac{\cos t^2}{2t}+1\right)$

and thus at $t=3$, the position $\vec{r}$ is $\boxed{(21, 1.152)}$

Work for problem 1(d)

Total distance can be given by the integral:

$$\int_0^3 \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt = \int_0^3 \sqrt{(4t+1)^2+(\sin(t^2))^2}\,dt \Rightarrow$$

$$\boxed{\Rightarrow 21.091}$$

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CALCULUS BC

SECTION II, Part A

Time—30 minutes

Number of problems—2

A graphing calculator is required for these problems.

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Work for problem 1(a)

$$\|v\| = \sqrt{[x'(t)]^2+[y'(t)]^2}$$
$$= \sqrt{x'(3)^2+y'(3)^2}$$
$$= \sqrt{169+\sin^2(9)}$$
$$\approx 13.006$$

$$a = \frac{1}{2}\left(x'(t)^2+y'(t)^2\right)^{-1/2}\left(2x'(t)x''(t)+2y'(t)y''(t)\right)$$
$$= \frac{1}{2}\left(\frac{1}{13}\right)\left(104+2\sin 9\cos 9\right)$$
$$= 3.913$$

Work for problem 1(b)

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\sin(t^2)}{4t+1}$$

$t=3$

$$\frac{\sin(9)}{13}$$

Continue problem 1 on page

Work for problem 1(c)

position $(x(t), y(t))$

$\frac{dx}{dt} = 4t+1$

$\int dx = \int (4t+1)\, dt$

$x(t) = 2t^2+t+C \qquad x(0)=0$

$0 = 0+0+C$

$C=0$

$x(t) = 2t^2+t$

$\frac{dy}{dt} = \sin(t^2)$

$\int dy = \int \sin(t^2) \cdot dt$

$y(t) = \frac{1}{2}\sin(t^2) +$

position $(21,$

Work for problem 1(d)

$\int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$

$\int_0^3 \sqrt{1+\left(\frac{dy/dt}{dx/dt}\right)^2}\, dx$

$= \int_0^3 \sqrt{1+\left(\frac{\sin t^2}{4t+1}\right)^2}\, dx$

$= 3.029$

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AP® CALCULUS BC
2011 SCORING COMMENTARY

Question 1

Overview

This problem described the path of a particle whose motion is described by $(x(t), y(t))$, where $x(t)$ and $y(t)$ satisfy $\frac{dx}{dt} = 4t + 1$ and $\frac{dy}{dt} = \sin\left(t^2\right)$ for $t \geq 0$. It is also given that $x(0) = 0$ and $y(0) = -4$. Part (a) asked for the speed of the particle at time $t = 3$ and the acceleration vector of the particle at $t = 3$. For part (b) students needed to recognize that the slope of the line tangent to the particle's path at $t = 3$ is given by $\left.\frac{dy}{dx}\right|_{t=3} = \left.\frac{dy/dt}{dx/dt}\right|_{t=3}$, a consequence of the chain rule. Part (c) asked for the position of the particle at time $t = 3$. This required two applications of the Fundamental Theorem: $x(3) = x(0) + \int_0^3 \frac{dx}{dt}\, dt$ and $y(3) = y(0) + \int_0^3 \frac{dy}{dt}\, dt$. Part (d) asked for the total distance traveled by the particle over the time interval $0 \leq t \leq 3$. This was found by integrating $\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$ over the interval $0 \leq t \leq 3$.

Sample: 1A
Score: 9

The student earned all 9 points.

Sample: 1B
Score: 6

The student earned 6 points: 1 point in part (a), 1 point in part (b), 2 points in part (c), and 2 points in part (d). In part (a) the student computes the speed but does not compute the acceleration vector. In parts (b) and (d) the student's work is correct. In part (c) the student finds $x(3)$ by solving the initial value problem and so the first 2 points were earned. The student's approach does not work for $y(3)$.

Sample: 1C
Score: 4

The student earned 4 points: 1 point in part (a), 1 point in part (b), 2 points in part (c), and no points in part (d). In part (a) the student computes the speed, but the acceleration vector is incorrect. In part (b) the student's work is correct. The answer is expressed in exact form as $\frac{\sin(9)}{13}$. In part (c) the student finds $x(3)$ by solving the initial value problem, so the first 2 points were earned. The student's approach does not work for $y(3)$. In part (d) the student presents an incorrect formula for total distance.