Question 5

Consider the differential equation $\frac{dy}{dx} = y^2(2x+2)$. Let $y = f(x)$ be the particular solution to the differential equation with initial condition $f(0) = -1$.

(a) Find $\lim_{x\to 0} \frac{f(x)+1}{\sin x}$. Show the work that leads to your answer.

(b) Use Euler's method, starting at $x = 0$ with two steps of equal size, to approximate $f\left(\frac{1}{2}\right)$.

(c) Find $y = f(x)$, the particular solution to the differential equation with initial condition $f(0) = -1$.

(a) $\lim_{x\to 0}(f(x)+1) = -1+1 = 0$ and $\lim_{x\to 0} \sin x = 0$

Using L'Hospital's Rule,

$$\lim_{x\to 0}\frac{f(x)+1}{\sin x} = \lim_{x\to 0}\frac{f'(x)}{\cos x} = \frac{f'(0)}{\cos 0} = \frac{(-1)^2\cdot 2}{1} = 2$$

$2 : \begin{cases} 1 : \text{L'Hospital's Rule} \\ 1 : \text{answer} \end{cases}$

(b) $$f\left(\frac{1}{4}\right) \approx f(0) + f'(0)\left(\frac{1}{4}\right) = -1 + (2)\left(\frac{1}{4}\right) = -\frac{1}{2}$$

$$f\left(\frac{1}{2}\right) \approx f\left(\frac{1}{4}\right) + f'\left(\frac{1}{4}\right)\left(\frac{1}{4}\right) = -\frac{1}{2} + \left(-\frac{1}{2}\right)^2\left(2\cdot\frac{1}{4}+2\right)\left(\frac{1}{4}\right) = -\frac{11}{32}$$

$2 : \begin{cases} 1 : \text{Euler's method} \\ 1 : \text{answer} \end{cases}$

(c) $$\frac{dy}{dx} = y^2(2x+2)$$

$$\frac{dy}{y^2} = (2x+2)\,dx$$

$$\int \frac{dy}{y^2} = \int (2x+2)\,dx$$

$$-\frac{1}{y} = x^2 + 2x + C$$

$$-\frac{1}{-1} = 0^2 + 2\cdot 0 + C \Rightarrow C = 1$$

$$-\frac{1}{y} = x^2 + 2x + 1$$

$$y = -\frac{1}{x^2+2x+1} = -\frac{1}{(x+1)^2}$$

Note: This solution is valid for $x > -1$.

$5 : \begin{cases} 1 : \text{separation of variables} \\ 1 : \text{antiderivatives} \\ 1 : \text{constant of integration} \\ 1 : \text{uses initial condition} \\ 1 : \text{solves for } y \end{cases}$

Note: max $2/5$ [1-1-0-0-0] if no constant of integration

Note: $0/5$ if no separation of variables

5A1

Do not write beyond this border.

5. Consider the differential equation $\frac{dy}{dx} = y^2(2x+2)$. Let $y = f(x)$ be the particular solution to the differential equation with initial condition $f(0) = -1$.

(a) Find $\lim_{x\to 0} \frac{f(x)+1}{\sin x}$. Show the work that leads to your answer.

a) $\lim_{x\to 0} \frac{f(x)+1}{\sin x} = \frac{-1+1}{0} = \frac{0}{0}$, Using L'Hôpital, $\lim_{x\to 0}\left(\frac{f(x)+1}{\sin x}\right) = \lim_{x\to 0} \frac{\frac{d}{dx}(f(x)+1)}{\frac{d}{dx}\sin x}$

indeterminate

$\lim_{x\to 0} \frac{f'(x)}{\cos x} = \lim_{x\to 0} \frac{y^2(2x+2)}{\cos x} = \frac{(-1)^2(2)}{1} = \boxed{2}$

(b) Use Euler's method, starting at $x = 0$ with two steps of equal size, to approximate $f\left(\frac{1}{2}\right)$.

$f(1/4) \approx f(0) + \frac{1}{4}f'(0) = -1 + \frac{1}{4}(2) = -1/2$

$f(1/2) \approx f(1/4) + \frac{1}{4}f'(1/4) = -1/2 + 1/4(1/4(1/2+2))$

$\boxed{f(1/2) \approx -1/2 + \frac{1}{4}\left(\frac{1}{4}(1/2+2)\right)}$

Continue problem 5 on page 19.

NO CALCULATOR ALLOWED

(c) Find $y = f(x)$, the particular solution to the differential equation with initial condition $f(0) = -1$.

$\frac{dy}{dx} = y^2(2x+2)$, $\frac{dy}{y^2} = (2x+2)dx$

$\int \frac{1}{y^2} dy = \int (2x+2) dx$, $-\frac{1}{y} = x^2 + 2x + C$

$-\frac{1}{-1} = 0^2 + 2(0) + C$, $C = 1$

$-\frac{1}{y} = x^2 + 2x + 1$

$\boxed{y = -\frac{1}{(x+1)^2}}$

Do not write beyond this border.

GO ON TO THE NEXT PAGE.

5B

NO CALCULATOR ALLOWED

5. Consider the differential equation $\frac{dy}{dx} = y^2(2x+2)$. Let $y = f(x)$ be the particular solution to the differential equation with initial condition $f(0) = -1$.

(a) Find $\lim_{x \to 0} \frac{f(x)+1}{\sin x}$. Show the work that leads to your answer.

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = f'(x)$$

$$\lim_{x \to 0} \frac{f(x)+1}{\sin(x)} = f'(\sin(x))$$

(b) Use Euler's method, starting at $x = 0$ with two steps of equal size, to approximate $f\left(\frac{1}{2}\right)$.

x	y	Δx	$\frac{dy}{dx}$ $y^2(2x+2)$	$\Delta y = \frac{dy}{dx} \cdot \Delta x$	$x+\Delta x$ new x	$y+\Delta y$ new y
0	-1	$\frac{1}{4}$	2	$\frac{1}{2}$	$\frac{1}{4}$	$-\frac{1}{2}$
$\frac{1}{4}$	$-\frac{1}{2}$	$\frac{1}{4}$	$\frac{5}{8}$	$\frac{5}{32}$	$\frac{1}{2}$	$-\frac{11}{32}$

$f\left(\frac{1}{2}\right) \approx -\frac{11}{32}$

$\left(-\frac{1}{2}\right)^2 \left(2 \cdot \frac{1}{4} + 2\right)$

$\frac{1}{2} + 2 =$

$\frac{1}{4} \cdot \frac{5}{2} = \frac{5}{8}$

$-\frac{1}{2} + \frac{5}{32} =$

$-\frac{16}{32} + \frac{5}{32} = -\frac{11}{32}$

Do not write beyond this border.

Continue problem 5 on page 19.

5B₂

Do not write beyond this border.

(c) Find $y = f(x)$, the particular solution to the differential equation with initial condition $f(0) = -1$.

$$\frac{dy}{dx} = y^2(2x+2)$$

$$\frac{dy}{y^2} = (2x+2)\,dx$$

$$\frac{-1}{y} = x^2 + 2x + C$$

$$\frac{-1}{-1} = \cancel{0}^2 + 20 + C \quad C = 1$$

$$y = x^2 + 2x + 1$$

Do not write beyond this border.

GO ON TO THE NEXT PAGE.

NO CALCULATOR ALLOWED

5C1

5. Consider the differential equation $\frac{dy}{dx} = y^2(2x+2)$. Let $y = f(x)$ be the particular solution to the differential equation with initial condition $f(0) = -1$.

(a) Find $\lim_{x\to 0} \frac{f(x)+1}{\sin x}$. Show the work that leads to your answer.

from part (C)

$$\lim_{x\to 0} \frac{\left(\sqrt{e^{(x^2+2x)}} - \frac{1}{\sqrt{e}}\right)+1}{\sin x}$$

$\lim_{x\to 0} \frac{f(x)}{g(x)} = \frac{0}{0}$ or und

$\lim_{x\to 0} \frac{f'(x)}{g'(x)}$

L'Hopital rule

$$\lim_{x\to 0} \frac{\left(e^{(x^2+2x)}\right)^{\frac{1}{2}} - e^{-\frac{1}{2}} + 1}{\sin x}$$

$$\lim_{x\to 0} = \frac{\frac{1}{2}(2x+2)\left(e^{(x^2+2x)}\right)^{-\frac{1}{2}} - e^{-\frac{1}{2}}}{\cos(x)} = \frac{\frac{2(1)}{2} - e^{-\frac{1}{2}}}{1}$$

$$\boxed{1 - e^{-\frac{1}{2}}}$$

(b) Use Euler's method, starting at $x = 0$ with two steps of equal size, to approximate $f\left(\frac{1}{2}\right)$.

$(0, -1)$

$\frac{dy}{dx} = y^2(2x+2)$

$y_n = y_{old} + f(x_{old}, y_{old}) \cdot \Delta x$

$\left(\frac{1}{4}\right)\left(\frac{5}{2}\right)$

$\frac{5}{8}$

$f(.25) = -1 + (2) \times (0.25)$

$f(.25) = -\frac{1}{2}$ $\quad (0.25, -\frac{1}{2})$

$f\left(\frac{1}{2}\right) = -\frac{1}{2} + \left(\frac{5}{8}\right) \times \left(\frac{1}{4}\right)$

$\boxed{f\left(\frac{1}{2}\right) = -\frac{11}{32}}$

$f\left(\frac{1}{2}\right) = -\frac{1}{2} + \frac{5}{32}$ $\quad f\left(\frac{1}{2}\right) = \frac{5}{32} - \frac{16}{32}$

Do not write beyond this border.

Do not write beyond this border.

Continue problem 5 on page 19.

5C2

(c) Find $y = f(x)$, the particular solution to the differential equation with initial condition $f(0) = -1$.

$$\left(\frac{dy}{dx} = \frac{y^2(2x+2)}{y^2}\right)dx$$

$$\int \frac{1}{y^2}\,dy = \int (2x+2)\,dx$$

$$\ln(y^2) = \frac{2x^2}{2} + 2x$$

$$\ln(y^2) = x^2 + 2x$$

$$\sqrt{y^2} = \sqrt{e^{x^2+2x}}$$

$$y = \sqrt{e^{(x^2+2x)}} + c$$

$$0 = \sqrt{e^{(-1)^2+2(-1)}} + c$$

$$0 = \sqrt{e^{1-2}} + c$$

$$0 = \sqrt{e^{-1}} + c$$

$$\boxed{f(x) = \sqrt{e^{(x^2+2x)}} - \frac{1}{\sqrt{e}}}$$

$$0 = \frac{1}{\sqrt{e}} + c$$

$$\boxed{c = -\frac{1}{\sqrt{e}}}$$

Do not write beyond this border.

Do not write beyond this border.

GO ON TO THE NEXT PAGE.

Question 5

Overview

This problem presented students with a differential equation and defined $y = f(x)$ to be the particular solution to the differential equation satisfying a given initial condition. Part (a) asked students to compute a limit of an expression involving $f(x)$, which required students to apply L'Hospital's Rule. Part (b) asked students to use Euler's method with two steps of equal size to approximate $f(x)$ at a value near the point given by the initial condition. Part (c) asked for the particular solution to the differential equation satisfying the given initial condition. Students should have used the method of separation of variables to solve the differential equation.

Sample: 5A
Score: 9

The student earned all 9 points.

Sample: 5B
Score: 6

The student earned 6 points: 2 points in part (b) and 4 points in part (c). The work in part (a) will not lead to an application of L'Hospital's Rule. In part (b) the student's work is correct. In part (c) the student has a correct separation of variables and the correct antiderivatives. The student correctly places the constant of integration and uses the initial condition. The student does not correctly solve for y and did not earn the fifth point.

Sample: 5C
Score: 3

The student earned 3 points: 2 points in part (b) and 1 point in part (c). In part (a) the student describes the conditions for using L'Hospital's Rule. The student imports an incorrect function from part (c). This function does not produce an indeterminate form; therefore, the application of L'Hospital's Rule is incorrect. The student did not earn any points in part (a). In part (b) the student's work is correct. In part (c) the student shows the separation of variables. The student incorrectly uses a logarithm when taking the antiderivative of $\frac{1}{y^2}$. The student did not earn the antiderivatives point or any of the remaining points.