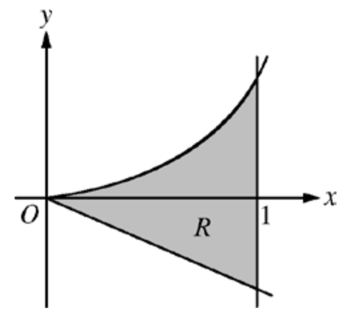
Question 5

Let R be the shaded region bounded by the graph of $y = xe^{x^2}$, the line $y = -2x$, and the vertical line $x = 1$, as shown in the figure above.



(a) Find the area of R.

(b) Write, but do not evaluate, an integral expression that gives the volume of the solid generated when R is rotated about the horizontal line $y = -2$.

(c) Write, but do not evaluate, an expression involving one or more integrals that gives the perimeter of R.

(a) $$\text{Area} = \int_0^1 \left(xe^{x^2} - (-2x)\right) dx = \left[\frac{1}{2}e^{x^2} + x^2\right]_{x=0}^{x=1} = \left(\frac{1}{2}e + 1\right) - \frac{1}{2} = \frac{e+1}{2}$$

$3 : \begin{cases} 1 : \text{integrand} \\ 1 : \text{antiderivative} \\ 1 : \text{answer} \end{cases}$

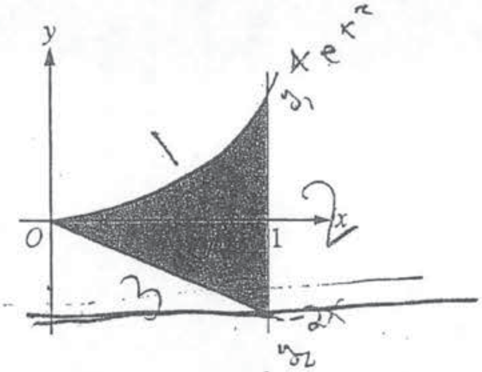
(b) $$\text{Volume} = \pi \int_0^1 \left[\left(xe^{x^2} + 2\right)^2 - (-2x + 2)^2\right] dx$$

$3 : \begin{cases} 2 : \text{integrand} \\ 1 : \text{limits and constant} \end{cases}$

(c) $$y' = \frac{d}{dx}\left(xe^{x^2}\right) = e^{x^2} + 2x^2e^{x^2} = e^{x^2}\left(1 + 2x^2\right)$$

$$\text{Perimeter} = \sqrt{5} + 2 + e + \int_0^1 \sqrt{1 + \left[e^{x^2}\left(1 + 2x^2\right)\right]^2}\, dx$$

$3 : \begin{cases} 1 : y' = e^{x^2}\left(1 + 2x^2\right) \\ 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$



5. Let R be the shaded region bounded by the graph of $y = xe^{x^2}$, the line $y = -2x$, and the vertical line $x = 1$, as shown in the figure above.

(a) Find the area of R.

$e^{x^2} + 2x^2e^{x^2}$

$$R = \int_0^1 x \cdot e^{x^2} dx - \int_0^1 -2x\, dx$$

$u = x^2, du = 2x$

$$= \frac{1}{2}\int_0^1 e^u du + 2\int_0^1 x\, dx$$

$$= \frac{1}{2} \cdot e^{x^2}\Big|_0^1 + x^2\Big|_0^1$$

$$= \frac{1}{2} \cdot e - \frac{1}{2} + 1 - 0$$

$$= \frac{e}{2} + \frac{1}{2} = \boxed{\frac{e+1}{2}}$$

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Continue problem 5 on page 19.

(b) Write, but do not evaluate, an integral expression that gives the volume of the solid generated when R is rotated about the horizontal line $y = -2$.

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$$\boxed{V = \pi \int_0^1 \left[(xe^{x^2} + 2)^2 - (-2x + 2)^2 \right] dx}$$

(c) Write, but do not evaluate, an expression involving one or more integrals that gives the perimeter of R.

Side 1: $\int_0^1 \sqrt{1 + (f'(x))^2}\, dx = \int_0^1 \sqrt{1 + (e^{x^2} + 2x^2 e^{x^2})^2}\, dx$

side 2: $y = 1 \cdot e^1 = e$, $y_2 = -2 \cdot 1 = -2$

length $= e + 2$

side 3: $\sqrt{a^2 + b^2} = \sqrt{1^2 + 2^2} = \sqrt{5}$

$$\boxed{P = S_1 + S_2 + S_3 = (e+2) + \sqrt{5} + \int_0^1 \sqrt{1 + (e^{x^2} + 2x^2 e^{x^2})^2}\, dx}$$

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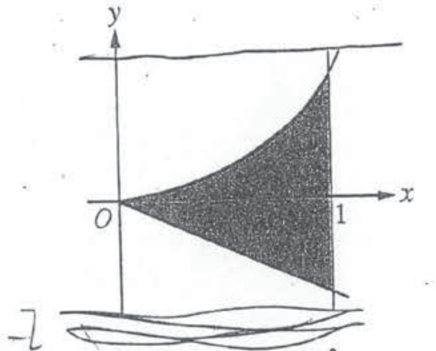

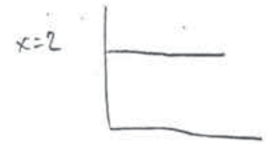

5. Let R be the shaded region bounded by the graph of $y = xe^{x^2}$, the line $y = -2x$, and the vertical line $x = 1$, as shown in the figure above.

(a) Find the area of R.

$\int_0^1 xe^{x^2}\,dx + \int_0^1 -2x\,dx$ = area R

$\frac{1}{2}e^{x^2}\Big|_0^1 \quad + -x^2\Big|_0^1$

$\left(\frac{1}{2}e^{1^2} - \frac{1}{2}e^{0^2}\right) + \left(-1^2 - (-0^2)\right)$ = area

Continue problem 5 on page 19.

NO CALCULATOR ALLOWED

(b) Write, but do not evaluate, an integral expression that gives the volume of the solid generated when R is rotated about the horizontal line $y = -2$.

$$\int_0^1 \left(\pi\left(xe^{x^2} - -2\right)^2 - \pi\left(-2x - -2\right)^2\right) dx$$

(c) Write, but do not evaluate, an expression involving one or more integrals that gives the perimeter of R.

$$\int_0^1 \sqrt{1 + \left(x(2xe^{x^2}) + e^{x^2}\right)^2} + \int_0^1 \sqrt{1 + (-2)^2}\, dx$$

$= \text{perimeter}$

$y' = x(2xe^{x^2}) + e^{x^2}$

$y' = -2$

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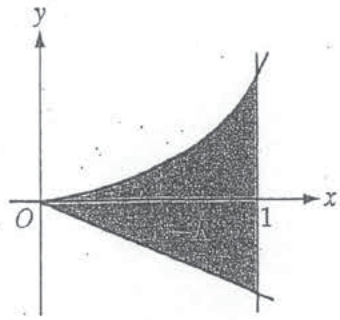

5. Let R be the shaded region bounded by the graph of $y = xe^{x^2}$, the line $y = -2x$, and the vertical line $x = 1$, as shown in the figure above.

(a) Find the area of R.

$$R = \underbrace{\left(\int_0^1 xe^{x^2}\,dx\right)} + \left|\tfrac{1}{2}(-2\cdot 1)\right|$$

$u = x^2$

$du = 2x\,dx$

$\tfrac{1}{2}du = x\,dx$

$$\int_0^1 \tfrac{1}{2}e^u\,du = \tfrac{1}{2}e^u \Big|_0^1$$

$$= \tfrac{1}{2}e^{x^2}\Big|_0^1$$

$$R = \left[\tfrac{1}{2}e^{1^2} - \tfrac{1}{2}e^{0^2}\right] + |-1| = \left[\tfrac{1}{2}e - 1\right] + 1$$

$$R = \tfrac{1}{2}e - 1 + 1 = \tfrac{1}{2}e$$

Continue problem 5 on page 19.

(b) Write, but do not evaluate, an integral expression that gives the volume of the solid generated when R is rotated about the horizontal line $y = -2$.

$$V = \int_0^1 \left[(xe^{x^2})^2 - (-2x)^2 \right] dx$$

(c) Write, but do not evaluate, an expression involving one or more integrals that gives the perimeter of R.

$$P = \left(\int_0^1 \sqrt{(2x^2e^{x^2} + e^{x^2}) + 1}\, dx \right) + \sqrt{(1)^2 + (2)^2} + (2 + e)$$

$$y' = x \cdot e^{x^2} \cdot 2x + e^{x^2} \cdot 1$$
$$= 2x^2e^{x^2} + e^{x^2}$$

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Question 5

Overview

In this problem students were given the graph of a region R bounded by the graph of $y = xe^{x^2}$, the line $y = -2x$, and the vertical line $x = 1$. In part (a) students were asked to find the area of R, requiring an appropriate integral setup and evaluation. Students needed to correctly evaluate $\int_0^1 \left(xe^{x^2} - (-2x)\right) dx$. Part (b) required students to find the volume of the solid generated when R is rotated about the horizontal line $y = -2$. Students needed to set up an integral where the integrand represents a cross-sectional area of a circular disc with inner radius $(-2x + 2)$ and outer radius $\left(xe^{x^2} + 2\right)$. This yielded the integral $\pi \int_0^1 \left(\left(xe^{x^2} + 2\right)^2 - (-2x + 2)^2\right) dx$. In part (c) students needed to write an expression involving one or more integrals that gives the perimeter of R. Students should have recognized that part of the perimeter involves finding the length of the curve $y = xe^{x^2}$ from $x = 0$ to $x = 1$ as well as the length of the two line segments. The resulting expression is $\int_0^1 \sqrt{1 + \left(e^{x^2}\left(1 + 2x^2\right)\right)^2}\, dx + \sqrt{5} + (2 + e)$.

Sample: 5A
Score: 9

The student earned all 9 points.

Sample: 5B
Score: 6

The student earned 6 points: 1 point in part (a), 3 points in part (b), and 2 points in part (c). In part (a) the student identifies the area of R as the sum of integrals from 0 to 1 of the given functions, rather than as the difference of these integrals. The student did not earn the first point. The student is eligible for and earned the second point for correctly antidifferentiating the stated integrands. The student is not eligible for the third point because both of the first 2 points were not earned. In part (b) the student's work is correct. In part (c) the student correctly differentiates xe^{x^2}, so the first point was earned. The student gives an integral equal to the length of the portion of the perimeter of R determined by xe^{x^2}, so the second point was earned. The student does not include the length of the vertical line segment in the expression for the perimeter of R, so the third point was not earned.

Question 5 (continued)

Sample: 5C
Score: 3

The student earned 3 points: 2 points in part (a), no points in part (b), and 1 point in part (c). In part (a) the student expresses the area of R as the sum of the integral from 0 to 1 of xe^{x^2} and a numeric value equal to the area of the triangular part of R below the x-axis. The student earned the first point. The student correctly antidifferentiates xe^{x^2}, so the second point was earned. The student makes an arithmetic error, so the third point was not earned. In part (b) the student does not present a correct integrand for the volume generated when R is rotated about the horizontal line $y = -2$. The student is not eligible for any points in part (b). In part (c) the student correctly differentiates xe^{x^2}, so the first point was earned. The student gives an integral that is not equal to the length of the portion of the perimeter of R determined by xe^{x^2} because the derivative of xe^{x^2} is not squared. The student did not earn the second point. The student is not eligible for the third point because both of the first 2 points were not earned.