Question 6

The Taylor series for a function f about $x = 1$ is given by $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2^n}{n}(x-1)^n$ and converges to $f(x)$ for $|x-1| < R$, where R is the radius of convergence of the Taylor series.

(a) Find the value of R.

(b) Find the first three nonzero terms and the general term of the Taylor series for f', the derivative of f, about $x = 1$.

(c) The Taylor series for f' about $x = 1$, found in part (b), is a geometric series. Find the function f' to which the series converges for $|x-1| < R$. Use this function to determine f for $|x-1| < R$.

(a) Let a_n be the nth term of the Taylor series.

$$\frac{a_{n+1}}{a_n} = \frac{(-1)^{n+2}\,2^{n+1}(x-1)^{n+1}}{n+1}\cdot\frac{n}{(-1)^{n+1}\,2^n(x-1)^n} = \frac{-2n(x-1)}{n+1}$$

$$\lim_{n\to\infty}\left|\frac{-2n(x-1)}{n+1}\right| = 2|x-1|$$

$$2|x-1| < 1 \Rightarrow |x-1| < \frac{1}{2}$$

The radius of convergence is $R = \frac{1}{2}$.

$3 : \begin{cases} 1 : \text{sets up ratio} \\ 1 : \text{computes limit of ratio} \\ 1 : \text{determines radius of convergence} \end{cases}$

(b) The first three nonzero terms are $2 - 4(x-1) + 8(x-1)^2$.

The general term is $(-1)^{n+1}2^n(x-1)^{n-1}$ for $n \ge 1$.

$3 : \begin{cases} 2 : \text{first three nonzero terms} \\ 1 : \text{general term} \end{cases}$

(c) The common ratio is $-2(x-1)$.

$$f'(x) = \frac{2}{1-(-2(x-1))} = \frac{2}{2x-1} \text{ for } |x-1| < \frac{1}{2}$$

$$f(x) = \int \frac{2}{2x-1}\,dx = \ln|2x-1| + C$$

$f(1) = 0$

$\ln|1| + C = 0 \Rightarrow C = 0$

$f(x) = \ln|2x-1|$ for $|x-1| < \frac{1}{2}$

$3 : \begin{cases} 1 : f'(x) \\ 1 : \text{antiderivative} \\ 1 : f(x) \end{cases}$

NO CALCULATOR ALLOWED

6. The Taylor series for a function f about $x = 1$ is given by $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{2^n}{n}(x-1)^n$ and converges to $f(x)$ for $|x - 1| < R$, where R is the radius of convergence of the Taylor series.

(a) Find the value of R.

This sum converges for x when $\lim_{n\to\infty} \left| \frac{\left(\frac{(-1)^{n+2} 2^{n+1} (x-1)^{n+1}}{n+1}\right)}{\left(\frac{(-1)^{n+1} 2^n (x-1)^n}{n}\right)} \right| < 1$

$\Rightarrow \lim_{n\to\infty} \left| \frac{(-1) \cdot 2 \cdot (x-1) \cdot n}{n+1} \right| < 1$

$\Rightarrow |2(x-1)| < 1$

So $|(x-1)| < \frac{1}{2}$

Thus $\boxed{R = \frac{1}{2}}$

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NO CALCULATOR ALLOWED

(b) Find the first three nonzero terms and the general term of the Taylor series for f', the derivative of f, about $x = 1$.

$$f(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} 2^n (x-1)^n}{n} \quad \text{for } |x-1| < R$$

$$\Rightarrow f'(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} 2^n \cdot n (x-1)^{n-1}}{n} = \sum_{n=1}^{\infty} (-1)^{n+1} 2^n \cdot (x-1)^{n-1}$$

The first three terms are: $2 - 4(x-1) + 8(x-1)^2$

(c) The Taylor series for f' about $x = 1$, found in part (b), is a geometric series. Find the function f' to which the series converges for $|x-1| < R$. Use this function to determine f for $|x-1| < R$.

$$f'(x) = 2\sum_{n=0}^{\infty} (-1)^n 2^n (x-1)^n = 2\sum_{n=0}^{\infty} (-2(x-1))^n$$

$$= 2 \cdot \frac{1}{1+2(x-1)} = \frac{2}{2x-1}$$

Then $f = \int f'$

$$= \int \frac{2}{2x-1}\,dx = \ln|2x-1| + C$$

$f(1) = 0$ because $\sum_{n=0}^{\infty} (-1)^{n+1} \frac{2^n}{n} (1-1)^n = 0$

$\ln|2(1)-1| + C = 0$

$\Rightarrow \quad C = 0$

$\boxed{\text{So } f(x) = \ln|2x-1| \text{ for } |x-1| < R}$

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6. The Taylor series for a function f about $x = 1$ is given by $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2^n}{n}(x-1)^n$ and converges to $f(x)$ for $|x-1| < R$, where R is the radius of convergence of the Taylor series.

(a) Find the value of R.

$$\lim_{n\to\infty}\left|\frac{\cancel{2^n}\,2\,\cancel{(x-1)^n}(x-1)}{(n+1)}\cdot\frac{(n)}{\cancel{(x-1)^n}\,\cancel{(2^n)}}\right| =$$

$$\lim_{n\to\infty}\left|\frac{2n(x-1)}{(n+1)}\right|$$

$$|x-1|\lim_{n\to\infty} 2 < 1$$

$$|x-1| < \boxed{\frac{1}{2}}$$

$$-\frac{1}{2} < x-1 < \frac{1}{2}$$

$$\boxed{R = \frac{1}{2}}$$

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6B₂

(b) Find the first three nonzero terms and the general term of the Taylor series for f', the derivative of f, about $x = 1$.

$$f = 2(x-1) - \frac{2^2}{2}(x-1)^2 + \frac{2^3}{3}(x-1)^3 \ldots (-1)^{n+1}\frac{2^n}{n}(x-1)^n$$

$$f' = 2 - \frac{2^2}{2}(2)(x-1) + \frac{2^3}{3}(3)(x-1)^2$$

$$\boxed{= 2 - 2^2(x-1) + 2^3(x-1)^2 \ldots (-1)^{n+1}2^n(x+1)^{n-1}}$$

(c) The Taylor series for f' about $x = 1$, found in part (b), is a geometric series. Find the function f' to which the series converges for $|x-1| < R$. Use this function to determine f for $|x-1| < R$.

$$\frac{a}{1-r} \qquad a = 2 \qquad r = \frac{1}{2(x-1)}$$

$$f' = \frac{2}{1 - \frac{1}{2(x-1)}} = \frac{2}{\frac{2(x-1)-1}{2(x-1)}} \boxed{= \frac{4(x-1)}{2(x-1)-1} = f'}$$

$$f = \frac{2(x-1)}{1 - \frac{1}{(x-1)}}$$

$$\cancel{\frac{2(x-1)}{\frac{2^2}{2}(x-1)^2}} \qquad \cancel{\frac{\frac{2^2}{2}(x-1)^2}{\frac{2^3}{3}(x-1)^3}}$$

$$f = \frac{2(x-1)}{\frac{(x-1)-1}{(x-1)}} = \boxed{\frac{2(x-1)^2}{(x-1)-1} = f}$$

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NO CALCULATOR ALLOWED

6. The Taylor series for a function f about $x = 1$ is given by $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2^n}{n}(x-1)^n$ and converges to $f(x)$ for $|x-1| < R$, where R is the radius of convergence of the Taylor series.

(a) Find the value of R.

$$\lim_{n\to\infty}\left|\frac{(-1)^{n+2}\frac{2^{n+1}}{n+1}(x-1)^{n+1}}{(-1)^{n+1}\frac{2^n}{n}(x-1)^n}\right|$$

$$\lim_{n\to\infty}\left|\frac{(-1)(2)(x-1)(n)}{(n+1)}\right|$$

$$\lim_{n\to\infty}\left|\frac{2n(x-1)}{n+1}\right| = 2$$

$$-2 < (x-1) < 2$$

$$-1 < (x-1) < 3 \qquad \frac{3+1}{2} = 2$$

$$\boxed{R = 2}$$

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NO CALCULATOR ALLOWED

(b) Find the first three nonzero terms and the general term of the Taylor series for f', the derivative of f, about $x = 1$.

$$\frac{2}{1}(x-1) - \frac{4}{2}(x-1)^2 + \frac{8}{3}(x-1)^3 \ldots + -1^{n+1}\frac{2^n}{n}(x-1)^n$$

$$\boxed{2 - 4(x-1) + 8(x-1)^2 + \ldots + (-1)^n \; 2n(x-1)^{n-1}}$$

(c) The Taylor series for f' about $x = 1$, found in part (b), is a geometric series. Find the function f' to which the series converges for $|x-1| < R$. Use this function to determine f for $|x-1| < R$.

$$\lim_{n\to\infty} \left| \frac{(-1)^{n+1}(2n+2)(x-1)^n}{(-1)^n(2n)(x-1)^{n-1}} \right|$$

$$\lim_{n\to\infty} \left| \frac{(-1)(2n+2)(x-1)}{2n} \right| = 1$$

$-1 < x-1 < 1$

$0 < x < 2$

$\frac{2}{2} = 1$

$\boxed{R = 1}$

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Question 6

Overview

In this problem students were given a Taylor series for a function f about $x = 1$. In part (a) students were asked to find the radius of convergence of this Taylor series. It was expected that students would use the ratio test to determine that the radius of convergence is $\frac{1}{2}$. In part (b) students needed to differentiate the series term-by-term to find the first three nonzero terms and the general term of the Taylor series for f'. In part (c) students were told that the Taylor series for f' is a geometric series. Students needed to know that finding the sum of that series requires dividing the first term of the series by the difference of 1 and the common ratio. This results in $f'(x) = \frac{2}{2x - 1}$. Students were also asked to find f. This required integrating $f'(x)$ to find $f(x) = \ln|2x - 1| + C$. In order to evaluate the constant of integration, students needed to use the initial condition that $f(1) = 0$ which yields $f(x) = \ln|2x - 1|$ for $|x - 1| < \frac{1}{2}$.

Sample: 6A
Score: 9

The student earned all 9 points. In part (c) the student does not need to qualify the closed form expression for $f(x)$ with "for $|x - 1| < R$."

Sample: 6B
Score: 6

The student earned 6 points: 3 points in part (a), 3 points in part (b), and no points in part (c). In parts (a) and (b), the student's work is correct. In part (b) the student writes the first three nonzero terms and the general term of the Taylor series of the original function f and then differentiates to find the required first three terms and the general term of the Taylor series of the derivative of f. Because the question asks students to find the first three nonzero terms and the general term of the Taylor series for f', the student is not penalized for omitting plus signs and an ellipsis in the boxed answer in part (b). In part (c) the student misidentifies the constant ratio in the geometric series. Because the first point was not earned, the student is not eligible for the third point. The student does not antidifferentiate correctly, so the second point was not earned.

Sample: 6C
Score: 3

The student earned 3 points: 1 point in part (a), 2 points in part (b), and no points in part (c). In part (a) the student earned 1 point with an appropriate ratio. The student's announced limit and conclusion are incorrect. In part (b) the student earned 2 points for giving the correct first three nonzero terms. The student's general term is incorrect. In part (c) the student does not provide an expression for f', so the student is not eligible for any points.