Question 6

The function f has a Taylor series about $x = 1$ that converges to $f(x)$ for all x in the interval of convergence. It is known that $f(1) = 1$, $f'(1) = -\frac{1}{2}$, and the nth derivative of f at $x = 1$ is given by $f^{(n)}(1) = (-1)^n \frac{(n-1)!}{2^n}$ for $n \geq 2$.

(a) Write the first four nonzero terms and the general term of the Taylor series for f about $x = 1$.

(b) The Taylor series for f about $x = 1$ has a radius of convergence of 2. Find the interval of convergence. Show the work that leads to your answer.

(c) The Taylor series for f about $x = 1$ can be used to represent $f(1.2)$ as an alternating series. Use the first three nonzero terms of the alternating series to approximate $f(1.2)$.

(d) Show that the approximation found in part (c) is within 0.001 of the exact value of $f(1.2)$.

(a) $f(1) = 1,\ f'(1) = -\frac{1}{2},\ f''(1) = \frac{1}{2^2},\ f'''(1) = -\frac{2}{2^3}$

$$f(x) = 1 - \frac{1}{2}(x-1) + \frac{1}{2^2 \cdot 2}(x-1)^2 - \frac{1}{2^3 \cdot 3}(x-1)^3 + \cdots + \frac{(-1)^n}{2^n \cdot n}(x-1)^n + \cdots$$

$4: \begin{cases} 1: \text{first two terms} \\ 1: \text{third term} \\ 1: \text{fourth term} \\ 1: \text{general term} \end{cases}$

(b) $R = 2$. The series converges on the interval $(-1, 3)$.

When $x = -1$, the series is $1 + 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$.
Since the harmonic series diverges, this series diverges.

When $x = 3$, the series is $1 - 1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \cdots$.
Since the alternating harmonic series converges, this series converges.

Therefore, the interval of convergence is $-1 < x \leq 3$.

$2: \begin{cases} 1: \text{identifies both endpoints} \\ 1: \text{analysis and interval of convergence} \end{cases}$

(c) $f(1.2) \approx 1 - \frac{1}{2}(0.2) + \frac{1}{8}(0.2)^2 = 1 - 0.1 + 0.005 = 0.905$

1 : approximation

(d) The series for $f(1.2)$ alternates with terms that decrease in magnitude to 0.

$$|f(1.2) - T_2(1.2)| \leq \left| \frac{-1}{2^3 \cdot 3}(0.2)^3 \right| = \frac{1}{3000} \leq 0.001$$

$2: \begin{cases} 1: \text{error form} \\ 1: \text{analysis} \end{cases}$

6. The function f has a Taylor series about $x = 1$ that converges to $f(x)$ for all x in the interval of convergence. It is known that $f(1) = 1$, $f'(1) = -\frac{1}{2}$, and the nth derivative of f at $x = 1$ is given by $f^{(n)}(1) = (-1)^n \frac{(n-1)!}{2^n}$ for $n \geq 2$.

(a) Write the first four nonzero terms and the general term of the Taylor series for f about $x = 1$.

$$1 + -\tfrac{1}{2}(x-1) + \frac{1}{2^2 \cdot 2!}(x-1)^2 - \frac{2!}{2^3 \cdot 3!}(x-1)^3 \dots + (-1)^n \frac{(n-1)!}{2^n}\frac{(x-1)^n}{n!}$$

$$1 - \frac{x-1}{2} + \frac{(x-1)^2}{8} - \frac{(x-1)^3}{24} \dots + (-1)^n \frac{(x-1)^n}{2^n \cdot n} + \dots$$

(b) The Taylor series for f about $x = 1$ has a radius of convergence of 2. Find the interval of convergence. Show the work that leads to your answer.

$$1 + \sum_{n=1}^{\infty} (-1)^n \frac{(x-1)^n}{2^n \cdot n}$$ converges over $x \in (-1, 3)$ (radius of convergence)

Endpoints:

$x = -1$ $\sum_{n=1}^{\infty} (-1)^n \frac{(-1-1)^n}{2^n \cdot n} = \sum_{n=1}^{\infty} (-1)^n (-1)^n \frac{2^n}{2^n} \cdot \frac{1}{n} = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges, (harmonic series)

$x = 3$ $\sum_{n=1}^{\infty} (-1)^n \frac{(3-1)^n}{2^n \cdot n} = \sum_{n=1}^{\infty} (-1)^n \frac{2^n}{2^n} \cdot \frac{1}{n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges (alternating harmonic series converges by AST)

Interval of convergence: $x \in (-1, 3]$

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(c) The Taylor series for f about $x = 1$ can be used to represent $f(1.2)$ as an alternating series. Use the first three nonzero terms of the alternating series to approximate $f(1.2)$.

$$1 - \frac{1.2-1}{2} + \frac{(1.2-1)^2}{8}$$

$$\frac{4}{800} = \frac{1}{200}$$

$$1 - .1 + \frac{.04}{8}$$

$$.9 + .005$$

$$.905 \approx f(1.2)$$

(d) Show that the approximation found in part (c) is within 0.001 of the exact value of $f(1.2)$.

In Alternating taylor series $|\text{error}| \leq |\text{next neglected term}|$

$$\frac{(1.2-1)^3}{24} \geq |\text{error}|$$

$$\frac{.008}{24} \geq |\text{error}|$$

$$\frac{.008}{24} \leq .001$$

$$|\text{error}| \leq .001$$

approx is within .001 of exact value.

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6. The function f has a Taylor series about $x = 1$ that converges to $f(x)$ for all x in the interval of convergence. It is known that $f(1) = 1$, $f'(1) = -\frac{1}{2}$, and the nth derivative of f at $x = 1$ is given by $f^{(n)}(1) = (-1)^n \frac{(n-1)!}{2^n}$ for $n \geq 2$.

(a) Write the first four nonzero terms and the general term of the Taylor series for f about $x = 1$.

$1 - \frac{1}{2}(x-1) + \frac{1}{4}$

$f^2(1) = \frac{1}{4}$

$f^3(1) = -\frac{1}{4}$

$f^4(1) = \frac{3}{8}$

$$1 - \frac{1}{2}(x-1) + \left(\frac{\frac{1}{4}(x-1)^2}{2!}\right) - \left(\frac{\frac{1}{4}(x-1)^3}{3!}\right) + \left(\frac{\frac{3}{8}(x-1)^4}{4!}\right)$$

(b) The Taylor series for f about $x = 1$ has a radius of convergence of 2. Find the interval of convergence. Show the work that leads to your answer.

$$r = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| =$$

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Continue problem 6 on page 21.

(c) The Taylor series for f about $x = 1$ can be used to represent $f(1.2)$ as an alternating series. Use the first three nonzero terms of the alternating series to approximate $f(1.2)$.

$$1-\frac{1}{2}(1.2-1)+\frac{\frac{1}{4}(1.2-1)^2}{2!}\approx f(1.2)$$

(d) Show that the approximation found in part (c) is within 0.001 of the exact value of $f(1.2)$.

$$\left|\frac{-\frac{1}{4}(1.2-1)^3}{3!}\right|<.001$$

error cannot be greater than the next term

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6. The function f has a Taylor series about $x = 1$ that converges to $f(x)$ for all x in the interval of convergence. It is known that $f(1) = 1$, $f'(1) = -\frac{1}{2}$, and the nth derivative of f at $x = 1$ is given by $f^{(n)}(1) = (-1)^n \frac{(n-1)!}{2^n}$ for $n \geq 2$.

(a) Write the first four nonzero terms and the general term of the Taylor series for f about $x = 1$.

$$f_{(4)}(x) = 1 - \tfrac{1}{2}(x-1) + \frac{\tfrac{1}{4}(x-1)^2}{2!} - \frac{\tfrac{1}{2}(x-1)^3}{3!}$$

$$f^{(2)}(1) = (-1)^2 \frac{(2-1)!}{2^2} = \tfrac{1}{4}$$

$$f^{(3)}(1) = (-1)^3 \frac{(3-1)!}{2^3} = \tfrac{4}{8} = \tfrac{1}{2}$$

(b) The Taylor series for f about $x = 1$ has a radius of convergence of 2. Find the interval of convergence. Show the work that leads to your answer.

$$p(x) = \sum (-1)^n \frac{(n-1)!}{2^n} (x-1)^n$$

$$\lim_{n \to \infty} \left| \frac{(n)!}{2^{n+1}} (x-1)^{n+1} \right| \cdot \left| \frac{2^n}{(n-1)!} \left(\frac{1}{(x-1)^n} \right) \right|$$

$$\lim_{n \to \infty} \left| \frac{n(x-1)}{2} \right| = \infty$$

$$(-\infty, \infty)$$

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(c) The Taylor series for f about $x = 1$ can be used to represent $f(1.2)$ as an alternating series. Use the first three nonzero terms of the alternating series to approximate $f(1.2)$.

$$f_{(3)}(x) = 1 - \frac{1}{2}(x-1) + \frac{\frac{1}{4}(x-1)^2}{2!}$$

$$\frac{0.04}{8}$$

$$f(1.2) = 1 - \frac{1}{2}(0.2) + \frac{1}{8}(0.2)^2$$

$$f(1.2) = 1 - 0.1 + 0.04/8$$

(d) Show that the approximation found in part (c) is within 0.001 of the exact value of $f(1.2)$.

$$|f(1.2) - 1.2| \leq 0.001$$

$$\left|1 - 0.1 + \frac{0.04}{8} - 1.2\right| \leq 0.001$$

$$\left|-0.3 + \frac{0.04}{8}\right| \leq 0.001$$

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Question 6

Overview

In this problem students were presented with a function f that has a Taylor series about $x = 1$ that converges to $f(x)$ for all x in the interval of convergence. The values of f, the first derivative of f, and nth derivatives of f at $x = 1$ are given. In part (a) students needed to write the first four nonzero terms and the general term of the Taylor series for f about $x = 1$. In part (b) students were given the radius of convergence and asked to find the interval of convergence. Since the series is centered about $x = 1$ with a radius of convergence of 2, students were expected to determine if the series converges at the endpoints $x = -1$ and $x = 3$. Students needed to use knowledge of the harmonic series and alternating harmonic series. In part (c) students were expected to use the Taylor series found in part (a) to represent $f(1.2)$ as an alternating series, and then use the first three nonzero terms of the alternating series to approximate $f(1.2)$. In part (d) students were expected to show that the approximation from part (c) is within 0.001 of the exact value of $f(1.2)$. The error of this approximation is bounded by the magnitude of the fourth term of the series for $f(1.2)$.

Sample: 6A
Score: 9

The response earned all 9 points. In part (a) the student presents the first four nonzero terms and a simplified version of the general term. In part (b) the student identifies the endpoints, substitutes the values into the series, correctly simplifies and identifies each series for the analysis, and presents the correct interval of convergence. In part (c) the student correctly substitutes into the first three terms and correctly evaluates to obtain the approximation. In part (d) the student correctly uses the next term as a bound on the error and compares the value to 0.001.

Sample: 6B
Score: 6

The response earned 6 points: 3 points in part (a), no points in part (b), 1 point in part (c), and 2 points in part (d). In part (a) the student correctly presents the first four nonzero terms of the series to earn the first 3 points. The student does not present a general term. In part (b) the student does not identify the correct endpoints and did not earn either point. Note that with a given radius of convergence R, it is not necessary for the student to use the ratio test to determine an interval for consideration. The student could have started this problem by considering $(1 - R, 1 + R)$. In part (c) the student's work is correct. The student uses the first three terms of the series to produce the correct approximation for $f(1.2)$. In part (d) the student correctly uses the next term as a bound on the error and compares the value to 0.001.

Sample: 6C
Score: 3

The response earned 3 points: 2 points in part (a), no points in part (b), 1 point in part (c), and no points in part (d). In part (a) the student correctly finds the first three nonzero terms of the series and earned the first 2 points. The student has an incorrect coefficient for the fourth term. The student does not present a general term. In part (b) the student does not identify the correct endpoints and did not earn either point. In part (c) the student's work is correct. The student uses the first three terms of the series to produce the correct approximation for $f(1.2)$. In part (d) the student does not present a correct error form.