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AP® CollegeBoard

# AP Calculus BC

## Sample Student Responses and Scoring Commentary

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**Question 6**

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(a) $f(0) = 0$

$f'(0) = 1$

$f''(0) = -1(1) = -1$

$f'''(0) = -2(-1) = 2$

$f^{(4)}(0) = -3(2) = -6$

The first four nonzero terms are

$$0 + 1x + \frac{-1}{2!}x^2 + \frac{2}{3!}x^3 + \frac{-6}{4!}x^4 = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}.$$

The general term is $\dfrac{(-1)^{n+1}x^n}{n}$.

$3 : \begin{cases} 1 : f''(0),\ f'''(0), \text{ and } f^{(4)}(0) \\ 1 : \text{verify terms} \\ 1 : \text{general term} \end{cases}$

(b) For $x = 1$, the Maclaurin series becomes $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$.

The series does not converge absolutely because the harmonic series diverges.

The series alternates with terms that decrease in magnitude to 0, and therefore the series converges conditionally.

2 : converges conditionally with reason

(c) $$\int_0^x f(t)\,dt = \int_0^x \left( t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \cdots + \frac{(-1)^{n+1}t^n}{n} + \cdots \right) dt$$

$$= \left[ \frac{t^2}{2} - \frac{t^3}{3 \cdot 2} + \frac{t^4}{4 \cdot 3} - \frac{t^5}{5 \cdot 4} + \cdots + \frac{(-1)^{n+1}t^{n+1}}{(n+1)n} + \cdots \right]_{t=0}^{t=x}$$

$$= \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{12} - \frac{x^5}{20} + \cdots + \frac{(-1)^{n+1}x^{n+1}}{(n+1)n} + \cdots$$

$3 : \begin{cases} 1 : \text{two terms} \\ 1 : \text{remaining terms} \\ 1 : \text{general term} \end{cases}$

(d) The terms alternate in sign and decrease in magnitude to 0. By the alternating series error bound, the error $\left| P_4\left(\frac{1}{2}\right) - g\left(\frac{1}{2}\right) \right|$ is bounded by the magnitude of the first unused term, $\left| -\dfrac{(1/2)^5}{20} \right|$.

Thus, $\left| P_4\left(\frac{1}{2}\right) - g\left(\frac{1}{2}\right) \right| \le \left| -\dfrac{(1/2)^5}{20} \right| = \dfrac{1}{32 \cdot 20} < \dfrac{1}{500}$.

1 : error bound

6A1

NO CALCULATOR ALLOWED

6A1

$$f(0) = 0$$
$$f'(0) = 1$$
$$f^{(n+1)}(0) = -n \cdot f^{(n)}(0) \text{ for all } n \geq 1$$

6. A function $f$ has derivatives of all orders for $-1 < x < 1$. The derivatives of $f$ satisfy the conditions above. The Maclaurin series for $f$ converges to $f(x)$ for $|x| < 1$.

(a) Show that the first four nonzero terms of the Maclaurin series for $f$ are $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}$, and write the general term of the Maclaurin series for $f$.

The Maclaurin series $P(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \cdots\cdots + \frac{f^{(n)}}{n!}x^n + \cdots$

$f(0)=0$, $f'(0)=1$, $f''(0) = -1$, $f'''(0) = 2$, $f^{(4)}(0) = -6$

$P_4(x) = 0 + \frac{1}{1}x + \frac{-1}{2}x^2 + \frac{2}{3!}x^3 + \frac{-6}{4!}x^4 = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}$

The $n$th term is: $\frac{f^{(n)}(0)}{n!}x^n = \frac{-(n-1)f^{(n-1)}(0)}{n!}x^n = \frac{(-1)^{n-1}x^n}{n}$

this is the general term: $\frac{(-1)^{n-1}x^n}{n}$

(b) Determine whether the Maclaurin series described in part (a) converges absolutely, converges conditionally, or diverges at $x = 1$. Explain your reasoning.

at $x=1$, the series is $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$;

① Because $\lim_{n\to\infty} \left|\frac{(-1)^{n-1}}{n}\right| = 0$, $\left|\frac{1}{n+1}\right| < \left|\frac{1}{n}\right|$ for all $n > 0$, by the alternating series convergence theorem, it converges.

② However, $\sum_{n=1}^{\infty} \left|\frac{(-1)^{n-1}}{n}\right| = \sum_{n=1}^{\infty} \frac{1}{n}$ is the known diverging harmonic series;

So the original Maclaurin series converges conditionally

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Continue question 6 on page 21.

(c) Write the first four nonzero terms and the general term of the Maclaurin series for $g(x) = \int_0^x f(t)\,dt$.

$g(x) = \int_0^x f(t)\,dt = \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{12}x^4 - \frac{1}{20}x^5 + \cdots + \frac{(-1)^n}{n(n-1)}x^n + \cdots$

$(g(0) = 0)$

(d) Let $P_n\left(\frac{1}{2}\right)$ represent the $n$th-degree Taylor polynomial for $g$ about $x = 0$ evaluated at $x = \frac{1}{2}$, where $g$ is the function defined in part (c). Use the alternating series error bound to show that

$\left|P_4\left(\frac{1}{2}\right) - g\left(\frac{1}{2}\right)\right| < \frac{1}{500}$.

By the alternating series error bound, the error $\epsilon_4(x)$, satisfies: $\epsilon_4(x) = |P_4(x) - g(x)| < b_5(x)$

where $b_5(x)$ is $\left|-\frac{1}{20}x^5\right|$, the absolute value of the 5th degree term in the Taylor polynomial.

at $x = \frac{1}{2}$, $b_5\left(\frac{1}{2}\right) = \frac{1}{20} \times \frac{1}{32} = \frac{1}{640} < \frac{1}{500}$

So $\left|P_4\left(\frac{1}{2}\right) - g\left(\frac{1}{2}\right)\right| = \epsilon_4(x) < \frac{1}{640} < \frac{1}{500}$.

$$f(0) = 0$$
$$f'(0) = 1$$
$$f^{(n+1)}(0) = -n \cdot f^{(n)}(0) \text{ for all } n \geq 1$$

6. A function $f$ has derivatives of all orders for $-1 < x < 1$. The derivatives of $f$ satisfy the conditions above. The Maclaurin series for $f$ converges to $f(x)$ for $|x| < 1$.

(a) Show that the first four nonzero terms of the Maclaurin series for $f$ are $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}$, and write the general term of the Maclaurin series for $f$.

general term

$\sum_{n=1}^{\infty} (-1)^{N+1} \frac{x^N}{N}$

Since $f(0)=0$

$f'(0)=1$

$f^{(2)}(0) = -1 \cdot f' = 1$

$f^{(3)}(0) =$

(b) Determine whether the Maclaurin series described in part (a) converges absolutely, converges conditionally, or diverges at $x = 1$. Explain your reasoning.

at $x=1$

$\sum_{n=1}^{\infty} (-1)^{N+1} \frac{(1)^N}{N} \Rightarrow \frac{(-1)^{N+1}}{N}$ = which is an alternating harmonic Series So it converges conditionally

Absolute Must have $\sum_{n=1}^{\infty} \frac{1}{N}$ coverge but $\frac{1}{N}$ is harmonic as $p \leq 1$ so diverges

$\frac{(-1)^{N+1}}{n+1} < \frac{(-1)^{N+1}}{N}$

$\lim_{n \to \infty} \frac{(-1)^{N+1}}{N} = 0$





Continue question 6 on page 21.

(c) Write the first four nonzero terms and the general term of the Maclaurin series for $g(x) = \int_0^x f(t)\,dt$.

$$\sum_{n=1}^{\infty} (-1)^{N+1} \frac{x^{N+1}}{n\cdot(n+1)}$$

Non zero term for $g(x) = \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{12} - \frac{x^5}{20}$

(d) Let $P_n\left(\frac{1}{2}\right)$ represent the $n$th-degree Taylor polynomial for $g$ about $x = 0$ evaluated at $x = \frac{1}{2}$, where $g$ is the function defined in part (c). Use the alternating series error bound to show that

$\left| P_4\left(\frac{1}{2}\right) - g\left(\frac{1}{2}\right) \right| < \frac{1}{500}$.

$$\left| P_4\left(\frac{1}{2}-\right) g\left(\frac{1}{2}\right) \right| < \frac{1}{500}$$

$$P_4(x) = \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{12}$$

$$P\left(\frac{1}{2}\right) = \frac{\frac{1}{4}}{2} - \frac{\frac{1}{8}}{6} + \frac{\frac{1}{16}}{12}$$

6C1

NO CALCULATOR ALLOWED

6C1

$$f(0) = 0$$
$$f'(0) = 1$$
$$f^{(n+1)}(0) = -n \cdot f^{(n)}(0) \text{ for all } n \geq 1$$

6. A function $f$ has derivatives of all orders for $-1 < x < 1$. The derivatives of $f$ satisfy the conditions above. The Maclaurin series for $f$ converges to $f(x)$ for $|x| < 1$.

(a) Show that the first four nonzero terms of the Maclaurin series for $f$ are $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}$, and write the general term of the Maclaurin series for $f$.

MacLaurin Series: $\frac{f^{(n)}(0) \cdot x^n}{n!}$

$n=1$ $\qquad n=2$ $\qquad n=3$ $\qquad n=4$

$\frac{f_1'(0) \cdot x^1}{1!}$ $\qquad \frac{-1 \cdot 1 x^2}{2!}$ $\qquad \frac{-2 \cdot -1 x^3}{3 \cdot 2}$ $\qquad \frac{-3 \cdot 2 x^4}{4 \cdot 3 \cdot 2}$

$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}$

(b) Determine whether the Maclaurin series described in part (a) converges absolutely, converges conditionally, or diverges at $x = 1$. Explain your reasoning.

$\frac{-n f^{(n)}(0) x^{n+1}}{(n+1)!}$ $\qquad \frac{-n f^{(n)}(0)}{(n+1)!}$

$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4}$

This series converges conditionally because when $x=1$ it can be compared to alternating harmonic, which converges conditionally



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6C2 **NO CALCULATOR ALLOWED** 6C2

(c) Write the first four nonzero terms and the general term of the Maclaurin series for $g(x) = \int_0^x f(t)\,dt$.

$$\frac{f^{(n)}(0)\,x^n}{n!}$$

$\cancel{n=1}$ n=2 n=3 n=4 n=5

$$\cancel{0} \quad \frac{x^2}{2} - \frac{x^3}{3} + \frac{x^4}{4} - \frac{x^5}{5}$$

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(d) Let $P_n\left(\frac{1}{2}\right)$ represent the $n$th-degree Taylor polynomial for $g$ about $x = 0$ evaluated at $x = \frac{1}{2}$, where $g$ is the function defined in part (c). Use the alternating series error bound to show that

$$\left|P_4\left(\frac{1}{2}\right) - g\left(\frac{1}{2}\right)\right| < \frac{1}{500}.$$

$$P_4\left(\tfrac{1}{2}\right) = \frac{\left(\frac{1}{2}\right)^5}{5} = \frac{\frac{1}{32}}{5} = \frac{1}{32\cdot 5}$$

$$\begin{array}{r} 1 \\ 32 \\ \times\ 5 \\ \hline 160 \end{array}$$

## Question 6

### Overview

In this problem students were presented with a function $f$ that has derivatives of all orders for $-1 < x < 1$ such that $f(0) = 0$, $f'(0) = 1$, and $f^{(n+1)}(0) = -n \cdot f^{(n)}(0)$ for all $n \geq 1$. It is also stated that the Maclaurin series for $f$ converges to $f(x)$ for $|x| < 1$. In part (a) students were asked to verify that the first four nonzero terms of the Maclaurin series for $f$ are $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}$ and to write the general term of this Maclaurin series. The $n$th-degree term of the Taylor polynomial for $f$ about $x = 0$ is $\frac{f^{(n)}(0)}{n!}x^n$. $f(0) = 0$ and $f'(0) = 1$ are given, and the given recurrence relation for $f^{(n+1)}(0)$ can be readily applied to see that $f''(0) = -1$, $f'''(0) = 2$, $f^{(4)}(0) = -6$, and $f^{(n)}(0) = (-1)^{n+1}(n-1)!$. Using these derivative values, students needed to confirm that the first four nonzero terms of the Maclaurin series for $f$ are as given, and that the general term is $\frac{(-1)^{n+1}x^n}{n}$. [LO 4.2A/EK 4.2A1] In part (b) students were asked to determine, with explanation, whether the Maclaurin series for $f$ converges absolutely, converges conditionally, or diverges at $x = 1$. Substituting $x = 1$, students should have obtained that the Maclaurin series for $f$ evaluated at $x = 1$ is $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$. Students needed to conclude that this series converges conditionally, noting that the series converges by the alternating series test and that $\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{n} \right|$ is the divergent harmonic series. [LO 4.1A/EK 4.1A4-4.1A6] In part (c) students were asked to find the first four nonzero terms and the general term of the Maclaurin series for $g(x) = \int_0^x f(t)\,dt$. Students needed to find these terms by integrating the Maclaurin series for $f$ term-by-term. [LO 4.2B/EK 4.2B5] In part (d) using the function $g$ defined in part (c), the expression $P_n\left(\frac{1}{2}\right)$ represents the $n$th-degree Taylor polynomial for $g$ about $x = 0$ evaluated at $x = \frac{1}{2}$. Students were directed to use the alternating series error bound to show that $\left| P_4\left(\frac{1}{2}\right) - g\left(\frac{1}{2}\right) \right| < \frac{1}{500}$. Students may have observed that the terms of the Taylor polynomial for $g$ about $x = 0$, evaluated at $x = \frac{1}{2}$, alternate in sign and decrease in magnitude to 0. Thus, the alternating series error bound can be applied to see that $\left| P_4\left(\frac{1}{2}\right) - g\left(\frac{1}{2}\right) \right| \leq \left| -\frac{(1/2)^5}{20} \right| = \frac{1}{32 \cdot 20}$, showing that the error in the approximation is less than $\frac{1}{500}$. [LO 4.1B/EK 4.1B2] This problem incorporates the following Mathematical Practices for AP Calculus (MPACs): reasoning with definitions and theorems, connecting concepts, implementing algebraic/computational processes, building notational fluency, and communicating.

## Question 6 (continued)

**Sample: 6A**
**Score: 9**

The response earned all 9 points: 3 points in part (a), 2 points in part (b), 3 points in part (c), and 1 point in part (d). In part (a) the student computes the numerical values of the derivatives in line 3 and earned the first point. The student uses the derivatives in line 4 to verify the given expression and earned the second point. The student produces a correct general term in line 5 using $-1$ with an exponent of $n-1$ rather than $n+1,$ which is still correct. The student earned the third point. In part (b) the student correctly uses the alternating series test to draw a conclusion of converges, identifies the harmonic series as divergent, and draws the correct conclusion of converges conditionally. The student earned both points. In part (c) the student produces the first four terms and earned the first 2 points. The student presents a correct general term using indices of $n$ rather than $n+1$ and earned the third point. In part (d) the student correctly computes the alternating series error and identifies it as the bound on the error.

**Sample: 6B**
**Score: 6**

The response earned 6 points: 1 point in part (a), 2 points in part (b), 3 points in part (c), and no point in part (d). In part (a) the student does not present the numerical values of the derivatives or include a proper verification. The student did not earn the first 2 points. The student produces a correct general term and earned the third point. The student is not penalized for using both lower and upper cases for $n$ in this question. In part (b) the student correctly identifies the series as the alternating harmonic and the absolute value series as the harmonic. The student draws the correct conclusion of converges conditionally. The student earned both points. In part (c) the student produces the first four terms and the general term. All 3 points were earned. In part (d) the student does not produce the numerical error value to verify that the error is less than $\frac{1}{500}.$ The student did not earn the point.

**Sample: 6C**
**Score: 3**

The response earned 3 points: 2 points in part (a), 1 point in part (b), no points in part (c), and no point in part (d). In part (a) the student computes the numerical values of the derivatives embedded in line 2 and earned the first point. The student's use of derivatives in the verification line is correct and earned the second point. The student does not present a general term, so the third point was not earned. In part (b) the student concludes that the alternating series converges conditionally, but the student does not explicitly address the series of absolute values of the terms. The student earned 1 of the 2 points. In part (c) the student produces four terms, but only the first term is correct. The student did not earn either of the first 2 points. The student does not produce a sufficient general term because line 1 is formulaic. The student did not earn the third point. In part (d) the student correctly computes the error bound based on the work in part (c) but does not connect $P_4\left(\frac{1}{2}\right) = \frac{1}{32 \cdot 5}$ to the error. The student did not earn the point.