2017

AP CollegeBoard

AP Calculus BC

Scoring Guidelines

Question 1

	1 : units in parts (a), (c), and (d)
(a) Volume $= \int_0^{10} A(h)\,dh$ $\approx (2-0)\cdot A(0) + (5-2)\cdot A(2) + (10-5)\cdot A(5)$ $= 2\cdot 50.3 + 3\cdot 14.4 + 5\cdot 6.5$ $= 176.3$ cubic feet	$2 : \begin{cases} 1 : \text{left Riemann sum} \\ 1 : \text{approximation} \end{cases}$
(b) The approximation in part (a) is an overestimate because a left Riemann sum is used and A is decreasing.	1 : overestimate with reason
(c) $\int_0^{10} f(h)\,dh = 101.325338$ The volume is 101.325 cubic feet.	$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$
(d) Using the model, $V(h) = \int_0^h f(x)\,dx$. $\left.\frac{dV}{dt}\right\|_{h=5} = \left[\frac{dV}{dh}\cdot\frac{dh}{dt}\right]_{h=5}$ $= \left[f(h)\cdot\frac{dh}{dt}\right]_{h=5}$ $= f(5)\cdot 0.26 = 1.694419$ When $h = 5,$ the volume of water is changing at a rate of 1.694 cubic feet per minute.	$3 : \begin{cases} 2 : \frac{dV}{dt} \\ 1 : \text{answer} \end{cases}$

Question 2

(a) $\frac{1}{2}\int_0^{\pi/2} (f(\theta))^2\, d\theta = 0.648414$

The area of R is 0.648.

$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$

(b) $\int_0^k \left((g(\theta))^2 - (f(\theta))^2\right) d\theta = \frac{1}{2}\int_0^{\pi/2} \left((g(\theta))^2 - (f(\theta))^2\right) d\theta$

— OR —

$\int_0^k \left((g(\theta))^2 - (f(\theta))^2\right) d\theta = \int_k^{\pi/2} \left((g(\theta))^2 - (f(\theta))^2\right) d\theta$

$2 : \begin{cases} 1 : \text{integral expression} \\ \quad\ \text{for one region} \\ 1 : \text{equation} \end{cases}$

(c) $w(\theta) = g(\theta) - f(\theta)$

$$w_A = \frac{\int_0^{\pi/2} w(\theta)\, d\theta}{\frac{\pi}{2} - 0} = 0.485446$$

The average value of $w(\theta)$ on the interval $\left[0, \frac{\pi}{2}\right]$ is 0.485.

$3 : \begin{cases} 1 : w(\theta) \\ 1 : \text{integral} \\ 1 : \text{average value} \end{cases}$

(d) $w(\theta) = w_A$ for $0 \le \theta \le \frac{\pi}{2} \Rightarrow \theta = 0.517688$

$w(\theta) = w_A$ at $\theta = 0.518$ (or 0.517).

$w'(0.518) < 0 \Rightarrow w(\theta)$ is decreasing at $\theta = 0.518$.

$2 : \begin{cases} 1 : \text{solves } w(\theta) = w_A \\ 1 : \text{answer with reason} \end{cases}$

Question 3

(a) $f(-6) = f(-2) + \int_{-2}^{-6} f'(x)\,dx = 7 - \int_{-6}^{-2} f'(x)\,dx = 7 - 4 = 3$ $f(5) = f(-2) + \int_{-2}^{5} f'(x)\,dx = 7 - 2\pi + 3 = 10 - 2\pi$	$3 : \begin{cases} 1 : \text{uses initial condition} \\ 1 : f(-6) \\ 1 : f(5) \end{cases}$
(b) $f'(x) > 0$ on the intervals $[-6, -2)$ and $(2, 5)$. Therefore, f is increasing on the intervals $[-6, -2]$ and $[2, 5]$.	2 : answer with justification
(c) The absolute minimum will occur at a critical point where $f'(x) = 0$ or at an endpoint. $f'(x) = 0 \Rightarrow x = -2, x = 2$	$2 : \begin{cases} 1 : \text{considers } x = 2 \\ 1 : \text{answer with justification} \end{cases}$

x	$f(x)$
-6	3
-2	7
2	$7 - 2\pi$
5	$10 - 2\pi$

The absolute minimum value is $f(2) = 7 - 2\pi$.

(d) $f''(-5) = \frac{2 - 0}{-6 - (-2)} = -\frac{1}{2}$ $\lim_{x\to 3^-} \frac{f'(x) - f'(3)}{x - 3} = 2$ and $\lim_{x\to 3^+} \frac{f'(x) - f'(3)}{x - 3} = -1$ $f''(3)$ does not exist because $\lim_{x\to 3^-} \frac{f'(x) - f'(3)}{x - 3} \neq \lim_{x\to 3^+} \frac{f'(x) - f'(3)}{x - 3}$.	$2 : \begin{cases} 1 : f''(-5) \\ 1 : f''(3) \text{ does not exist,} \\ \quad\ \text{with explanation} \end{cases}$

Question 4

(a) $H'(0) = -\frac{1}{4}(91 - 27) = -16$

$H(0) = 91$

An equation for the tangent line is $y = 91 - 16t$.

The internal temperature of the potato at time $t = 3$ minutes is approximately $91 - 16 \cdot 3 = 43$ degrees Celsius.

$$3 : \begin{cases} 1 : \text{slope} \\ 1 : \text{tangent line} \\ 1 : \text{approximation} \end{cases}$$

(b) $\frac{d^2H}{dt^2} = -\frac{1}{4}\frac{dH}{dt} = \left(-\frac{1}{4}\right)\left(-\frac{1}{4}\right)(H - 27) = \frac{1}{16}(H - 27)$

$H > 27$ for $t > 0 \Rightarrow \frac{d^2H}{dt^2} = \frac{1}{16}(H - 27) > 0$ for $t > 0$

Therefore, the graph of H is concave up for $t > 0$. Thus, the answer in part (a) is an underestimate.

1 : underestimate with reason

(c) $\frac{dG}{(G - 27)^{2/3}} = -dt$

$\int \frac{dG}{(G - 27)^{2/3}} = \int (-1)\, dt$

$3(G - 27)^{1/3} = -t + C$

$3(91 - 27)^{1/3} = 0 + C \Rightarrow C = 12$

$3(G - 27)^{1/3} = 12 - t$

$G(t) = 27 + \left(\frac{12 - t}{3}\right)^3$ for $0 \le t < 10$

The internal temperature of the potato at time $t = 3$ minutes is $27 + \left(\frac{12 - 3}{3}\right)^3 = 54$ degrees Celsius.

$$5 : \begin{cases} 1 : \text{separation of variables} \\ 1 : \text{antiderivatives} \\ 1 : \text{constant of integration and} \\ \quad\ \text{uses initial condition} \\ 1 : \text{equation involving } G \text{ and } t \\ 1 : G(t) \text{ and } G(3) \end{cases}$$

Note: max $2/5$ [1-1-0-0-0] if no constant of integration

Note: $0/5$ if no separation of variables

Question 5

(a) $f'(x) = \dfrac{-3(4x-7)}{\left(2x^2 - 7x + 5\right)^2}$

$f'(3) = \dfrac{(-3)(5)}{(18 - 21 + 5)^2} = -\dfrac{15}{4}$

2 : $f'(3)$

(b) $f'(x) = \dfrac{-3(4x-7)}{\left(2x^2 - 7x + 5\right)^2} = 0 \Rightarrow x = \dfrac{7}{4}$

The only critical point in the interval $1 < x < 2.5$ has x-coordinate $\dfrac{7}{4}$.

f' changes sign from positive to negative at $x = \dfrac{7}{4}$.

Therefore, f has a relative maximum at $x = \dfrac{7}{4}$.

2 : { 1 : x-coordinate
1 : relative maximum with justification

(c) $\displaystyle\int_5^\infty f(x)\,dx = \lim_{b\to\infty}\int_5^b \frac{3}{2x^2 - 7x + 5}\,dx = \lim_{b\to\infty}\int_5^b \left(\frac{2}{2x-5} - \frac{1}{x-1}\right)dx$

$\displaystyle = \lim_{b\to\infty}\Big[\ln(2x-5) - \ln(x-1)\Big]_5^b = \lim_{b\to\infty}\left[\ln\left(\frac{2x-5}{x-1}\right)\right]_5^b$

$\displaystyle = \lim_{b\to\infty}\left[\ln\left(\frac{2b-5}{b-1}\right) - \ln\left(\frac{5}{4}\right)\right] = \ln 2 - \ln\left(\frac{5}{4}\right) = \ln\left(\frac{8}{5}\right)$

3 : { 1 : antiderivative
1 : limit expression
1 : answer

(d) f is continuous, positive, and decreasing on $[5, \infty)$.

The series converges by the integral test since $\displaystyle\int_5^\infty \frac{3}{2x^2 - 7x + 5}\,dx$ converges.

— OR —

$\dfrac{3}{2n^2 - 7n + 5} > 0$ and $\dfrac{1}{n^2} > 0$ for $n \ge 5$.

Since $\displaystyle\lim_{n\to\infty}\frac{\frac{3}{2n^2 - 7n + 5}}{\frac{1}{n^2}} = \frac{3}{2}$ and the series $\displaystyle\sum_{n=5}^{\infty}\frac{1}{n^2}$ converges,

the series $\displaystyle\sum_{n=5}^{\infty}\frac{3}{2n^2 - 7n + 5}$ converges by the limit comparison test.

2 : answer with conditions

Question 6

(a) $f(0) = 0$

$f'(0) = 1$

$f''(0) = -1(1) \quad = -1$

$f'''(0) = -2(-1) = 2$

$f^{(4)}(0) = -3(2) = -6$

The first four nonzero terms are

$$0 + 1x + \frac{-1}{2!}x^2 + \frac{2}{3!}x^3 + \frac{-6}{4!}x^4 = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}.$$

The general term is $\dfrac{(-1)^{n+1}x^n}{n}$.

$3 : \begin{cases} 1 : f''(0),\ f'''(0),\ \text{and}\ f^{(4)}(0) \\ 1 : \text{verify terms} \\ 1 : \text{general term} \end{cases}$

(b) For $x = 1$, the Maclaurin series becomes $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$.

The series does not converge absolutely because the harmonic series diverges.

The series alternates with terms that decrease in magnitude to 0, and therefore the series converges conditionally.

2 : converges conditionally with reason

(c)
$$\int_0^x f(t)\,dt = \int_0^x \left(t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \cdots + \frac{(-1)^{n+1}t^n}{n} + \cdots \right) dt$$
$$= \left[\frac{t^2}{2} - \frac{t^3}{3\cdot 2} + \frac{t^4}{4\cdot 3} - \frac{t^5}{5\cdot 4} + \cdots + \frac{(-1)^{n+1}t^{n+1}}{(n+1)n} + \cdots \right]_{t=0}^{t=x}$$
$$= \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{12} - \frac{x^5}{20} + \cdots + \frac{(-1)^{n+1}x^{n+1}}{(n+1)n} + \cdots$$

$3 : \begin{cases} 1 : \text{two terms} \\ 1 : \text{remaining terms} \\ 1 : \text{general term} \end{cases}$

(d) The terms alternate in sign and decrease in magnitude to 0. By the alternating series error bound, the error $\left| P_4\left(\frac{1}{2}\right) - g\left(\frac{1}{2}\right) \right|$ is bounded by the magnitude of the first unused term, $\left| -\dfrac{(1/2)^5}{20} \right|$.

Thus, $\left| P_4\left(\frac{1}{2}\right) - g\left(\frac{1}{2}\right) \right| \le \left| -\dfrac{(1/2)^5}{20} \right| = \dfrac{1}{32\cdot 20} < \dfrac{1}{500}$.

1 : error bound