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AP CollegeBoard

AP Calculus BC

Sample Student Responses and Scoring Commentary

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Question 6

(a) The first four nonzero terms are $\frac{x^2}{3} - \frac{x^3}{2 \cdot 3^2} + \frac{x^4}{3 \cdot 3^3} - \frac{x^5}{4 \cdot 3^4}$.

The general term is $(-1)^{n+1} \frac{x^{n+1}}{n \cdot 3^n}$.

$2 : \begin{cases} 1 : \text{first four terms} \\ 1 : \text{general term} \end{cases}$

(b) $$\lim_{n\to\infty} \left| \frac{\dfrac{(-1)^{n+2} x^{n+2}}{(n+1)\left(3^{n+1}\right)}}{\dfrac{(-1)^{n+1} x^{n+1}}{n \cdot 3^n}} \right| = \lim_{n\to\infty} \left| \frac{-x}{3} \cdot \frac{n}{(n+1)} \right| = \left| \frac{x}{3} \right|$$

$\left|\frac{x}{3}\right| < 1$ for $|x| < 3$

Therefore, the radius of convergence of the Maclaurin series for f is 3.

— OR —

The radius of convergence of the Maclaurin series for $\ln(1 + x)$ is 1, so the series for $f(x) = x \ln\left(1 + \frac{x}{3}\right)$ converges absolutely for $\left|\frac{x}{3}\right| < 1$.

$\left|\frac{x}{3}\right| < 1 \Rightarrow |x| < 3$

Therefore, the radius of convergence of the Maclaurin series for f is 3.

When $x = -3$, the series is $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{(-3)^{n+1}}{n \cdot 3^n} = \sum_{n=1}^{\infty} \frac{3}{n}$, which diverges by comparison to the harmonic series.

When $x = 3$, the series is $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{3^{n+1}}{n \cdot 3^n} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{3}{n}$, which converges by the alternating series test.

The interval of convergence of the Maclaurin series for f is $-3 < x \le 3$.

$5 : \begin{cases} 1 : \text{sets up ratio} \\ 1 : \text{computes limit of ratio} \\ 1 : \text{radius of convergence} \\ 1 : \text{considers both endpoints} \\ 1 : \text{analysis and interval of convergence} \end{cases}$

— OR —

$5 : \begin{cases} 1 : \text{radius for } \ln(1 + x) \text{ series} \\ 1 : \text{substitutes } \frac{x}{3} \\ 1 : \text{radius of convergence} \\ 1 : \text{considers both endpoints} \\ 1 : \text{analysis and interval of convergence} \end{cases}$

(c) By the alternating series error bound, an upper bound for $|P_4(2) - f(2)|$ is the magnitude of the next term of the alternating series.

$|P_4(2) - f(2)| < \left| -\frac{2^5}{4 \cdot 3^4} \right| = \frac{8}{81}$

$2 : \begin{cases} 1 : \text{uses fifth-degree term as error bound} \\ 1 : \text{answer} \end{cases}$

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6. The Maclaurin series for $\ln(1 + x)$ is given by

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^{n+1}\frac{x^n}{n} + \cdots.$$

On its interval of convergence, this series converges to $\ln(1 + x)$. Let f be the function defined by $f(x) = x\ln\left(1 + \frac{x}{3}\right)$.

(a) Write the first four nonzero terms and the general term of the Maclaurin series for f.

$$f(x) \approx \frac{x^2}{3} - \frac{x\left(\frac{x}{3}\right)^2}{2} + \frac{x\left(\frac{x}{3}\right)^3}{3} - \frac{x\left(\frac{x}{3}\right)^4}{4}$$

$$\approx \frac{x^2}{3} - \frac{\frac{x^3}{9}}{2} + \frac{\frac{x^4}{27}}{3} - \frac{\frac{x^5}{81}}{4} + \ldots + (-1)^{n+1}\frac{\frac{x^{n+1}}{3^n}}{n}$$

$$\approx \frac{x^2}{3} - \frac{x^3}{9\cdot 2} + \frac{x^4}{27\cdot 3} - \frac{x^5}{81\cdot 4} + \ldots + (-1)^{n+1}\frac{x^{n+1}}{n3^n}$$

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6A 2*2

(b) Determine the interval of convergence of the Maclaurin series for f. Show the work that leads to your answer.

Ratio Test: $\lim_{n\to\infty}\left|\frac{(-1)^{n+2}x^{n+2}}{(n+1)3^{n+1}}\cdot\frac{n3^n}{(-1)^{n+1}x^{n+1}}\right|$

$\lim_{n\to\infty}\left|\frac{-nx}{3(n+1)}\right|$

$\left|\frac{x}{3}\right|\lim_{n\to\infty}\left|\frac{n}{n+1}\right|$

$\left|\frac{x}{3}\right|\cdot 1 < 1$

$\left|\frac{x}{3}\right| < 1$

$-1 < \frac{x}{3} < 1$

$-3 < x < 3$

$3: \quad \frac{(-1)^{n+1}\cdot 3^{n+1}}{n3^n} = \frac{3(-1)^{n+1}}{n}$

alternating harmonic
∴ converges
conditionally

$-3: \quad \frac{(-1)^{n+1}(-3)^{n+1}}{n3^n} = \frac{(-1)^{2n+2}(3)^{n+1}}{n3^n}$

$= \frac{3}{n}$ harmonic
∴ diverges

Interval: $-3 < x \le 3$

(c) Let $P_4(x)$ be the fourth-degree Taylor polynomial for f about $x = 0$. Use the alternating series error bound to find an upper bound for $|P_4(2) - f(2)|$.

$|P_4(2) - f(2)| <$

$= \left|\frac{x^5}{81\cdot 4}\right|$

$= \left|\frac{-2^5}{81\cdot 4}\right|$

$\frac{32}{81\cdot 4} = \frac{8}{81}$

$|P_4(2) - f(2)| < \frac{8}{81}$

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6. The Maclaurin series for $\ln(1+x)$ is given by

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^{n+1}\frac{x^n}{n} + \cdots.$$

On its interval of convergence, this series converges to $\ln(1+x)$. Let f be the function defined by $f(x) = x\ln\left(1+\frac{x}{3}\right)$.

(a) Write the first four nonzero terms and the general term of the Maclaurin series for f.

$$x\ln(1+x) = x^2 - \frac{x^3}{2} + \frac{x^4}{3} - \frac{x^5}{4} + \ldots + \frac{(-1)^{n+1}x^{n+1}}{n}$$

$$x\ln\left(1+\frac{x}{3}\right) = \left(\frac{x}{3}\right)^2 - \frac{1}{2}\left(\frac{x}{3}\right)^3 + \frac{1}{3}\left(\frac{x}{3}\right)^4 - \frac{1}{4}\left(\frac{x}{3}\right)^5$$

$$= \boxed{\frac{x^2}{9} - \frac{x^3}{2\cdot 27} + \frac{x^4}{3\cdot 81} - \frac{x^5}{4\cdot 243} + \ldots + \frac{(-1)^{n+1}\left(\frac{x}{3}\right)^{n+1}}{n}}$$

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(b) Determine the interval of convergence of the Maclaurin series for f. Show the work that leads to your answer.

$$f(x) = x\ln\left(1+\frac{x}{3}\right)$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+2}\left(\frac{x}{3}\right)^{n+2}}{n+1}\cdot\frac{n}{\cancel{(-1)^{n+1}}\cdot\left(\frac{x}{3}\right)^{n+1}}\right| = \left|\frac{x}{3}\right|$$

$\left|\frac{x}{3}\right| < 1$ $\quad |x| < 3$ $\quad x = 3$

$x = -3$

$$\boxed{-3 < x \leq 3}$$

$x = 3$ $\quad \frac{(-1)^{n+1}(1)^{n+1}}{n+1}$ converges

$x = -3$ $\quad \frac{(-1)^{n+1}(-1)^{n+1}}{n+1}$ diverges

(c) Let $P_4(x)$ be the fourth-degree Taylor polynomial for f about $x = 0$. Use the alternating series error bound to find an upper bound for $|P_4(2) - f(2)|$.

$$|P_4(2) - f(2)| < \left|\frac{-(2)^5}{4\cdot 243}\right|$$

$$\boxed{\text{upper bound is } \frac{2^5}{4\cdot 243}}$$

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NO CALCULATOR ALLOWED

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6. The Maclaurin series for $\ln(1+x)$ is given by

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^{n+1}\frac{x^n}{n} + \cdots.$$

On its interval of convergence, this series converges to $\ln(1+x)$. Let f be the function defined by $f(x) = x\ln\left(1 + \frac{x}{3}\right)$.

(a) Write the first four nonzero terms and the general term of the Maclaurin series for f. H

$$f(x) = \frac{x^2}{3} - \frac{x^3}{6} + \frac{x^4}{9} - \frac{x^5}{12} + \ldots + \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{n+1}}{3n}$$

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(b) Determine the interval of convergence of the Maclaurin series for f. Show the work that leads to your answer.

$$\lim_{n\to\infty}\left|\frac{\frac{(-1)^{n+2}x^{n+2}}{3n+3}}{\frac{(-1)^{n+1}(x)^{n+1}}{3n}}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{\cancel{n+2}}(\cancel{x^{n+2}})\cdot(3n)}{\cancel{(-1)^{n+1}}(\cancel{(x)^{n+1}})(3n+3)}\right|$$

$$= \lim_{n\to\infty}\left|\frac{-x\,3n}{3n+3}\right| = |-x|$$

$$|-x| < 1$$

$$\boxed{-1 < x \leq 1}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(-1)^{n+1}}{3n}$$ harmonic diverges

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\cancel{(1)^{n+1}}}{3n}$$ converges alt. harmonic

(c) Let $P_4(x)$ be the fourth-degree Taylor polynomial for f about $x = 0$. Use the alternating series error bound to find an upper bound for $|P_4(2) - f(2)|$.

$$\left|-\frac{x^4}{4} - \frac{x^5}{12}\right| \leq \frac{x^6}{15}$$

$$\left|-4 - \frac{8}{3}\right| \leq \frac{64}{15}$$

$$\left|\frac{-16}{3}\right| \leq \frac{64}{15}$$

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Question 6

Overview

In this problem the first four nonzero terms and the general term of the Maclaurin series for $\ln(1 + x)$ are given, and the function f is defined by $f(x) = x\ln\left(1 + \frac{x}{3}\right)$. In part (a) students were asked for the first four nonzero terms and the general term of the Maclaurin series for f. A correct response should substitute $\frac{x}{3}$ for x in the supplied terms of the series for $\ln(1 + x)$, multiply the resulting terms by x, and expand so that each term is a constant multiple of a power of x. The general term should also be included. In part (b) students were asked to determine the interval of convergence of the Maclaurin series for f with supporting work for their answer. A correct response should demonstrate the use of the ratio test to determine the radius of convergence of the series and, then, a test of the endpoints of the interval of convergence to determine which endpoints, if any, are to be included in the interval of convergence. In part (c) students were asked to use the alternating series error bound to find an upper bound for $|P_4(2) - f(2)|$, where $P_4(x)$ is the fourth-degree Taylor polynomial for f about $x = 0$. A correct response should indicate that the alternating series error bound bounds $|P_4(2) - f(2)|$ by the magnitude of the next term in the alternating series formed by evaluating the Taylor series for f about $x = 0$ at $x = 2$.

For part (a) see LO 4.2B/EK 4.2B5. For part (b) see LO 4.1A/EK 4.1A3, LO 4.1A/EK 4.1A6, LO 4.2C/EK 4.2C2. For part (c) see LO 4.2A/EK 4.2A5. This problem incorporates the following Mathematical Practices for AP Calculus (MPACs): reasoning with definitions and theorems, connecting concepts, implementing algebraic/computational processes, building notational fluency, and communicating.

Sample: 6A
Score: 9

The response earned all 9 points: 2 points in part (a), 5 points in part (b), and 2 points in part (c). In part (a) the response would have earned the first point in line 2 for the first four nonzero terms of the Maclaurin series for f with no simplification. In this case, the simplification is correct, and the point was earned in line 3. The response would have earned the second point in line 2 with the general term of the Maclaurin series for f with no simplification. In this case, the simplification is correct, and the point was earned in line 3. In part (b) the response earned the first point with the ratio in line 1 on the left. The response earned the second point in line 4 on the left with $\left|\frac{x}{3}\right|$. The response earned the third point in line 7 on the left with $-3 < x < 3$. The response earned the fourth point with the work on the right by substituting the endpoints $x = 3$ and $x = -3$ into the general term of the Maclaurin series for f. The response earned the fifth point for the work on the right with the analysis — for $x = 3$ stating "alternating harmonic $\therefore$ converges conditionally" and for $x = -3$ stating "harmonic $\therefore$ diverges" — and the interval of convergence in the last line on the right. In part (c) the response earned the first point in line 1 on the right with use of the fifth-degree term. The response earned the second point by substituting 2 for x in the fifth-degree term and taking an absolute value to guarantee that a positive error bound is returned.

Sample: 6B
Score: 6

The response earned 6 points: no points in part (a), 4 points in part (b), and 2 points in part (c). In part (a) the response did not earn the first point as the denominators are incorrect in all of the first four nonzero terms of the Maclaurin series for f. The response did not earn the second point because of an incorrect general term of the Maclaurin series for f. In part (b) the response earned the first point for a ratio in line 2 that is consistent with the

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Question 6 (continued)

general term presented in part (a). The response earned the second point in line 2 with $\left|\frac{x}{3}\right|$. The response earned the third point with $|x| < 3$ in line 3 and a radius of convergence of the Maclaurin series for f that is consistent with the general term presented in part (a). The response earned the fourth point by substituting the endpoints $x = 3$ and $x = -3$ into the imported general term from part (a). The response did not earn the fifth point as the response states "converges" and "diverges" without justification. In part (c) the response earned both points with an expression for the absolute value of the fifth-degree term evaluated at $x = 2$ that is consistent with the fifth-degree term of the series in part (a).

Sample: 6C
Score: 3

The response earned 3 points: no points in part (a), 3 points in part (b), and no points in part (c). In part (a) the response did not earn the first point as the response has incorrect denominators in the second, third, and fourth terms of the Maclaurin series for f. The response did not earn the second point because of an incorrect general term of the Maclaurin series for f. In part (b) the response earned the first point for a ratio given in the top left corner that is consistent with the general term presented in part (a). The incorrect general term imported from part (a) results in an oversimplification of the question, so the response is not eligible for the second and third points. The response earned the fourth point by substituting $x = -1$ and $x = 1$ into the general term imported from part (a). The response earned the fifth point with the analysis — for $x = -1$ with "harmonic diverges" and for $x = 1$ with "converges alt. harmonic"— and the interval of convergence boxed on the left that is consistent with the general term presented in part (a). In part (c) the response did not earn the first point because the sixth-degree term is used to find the requested upper bound instead of the fifth-degree term. The response did not earn the second point because the error bound is not consistent with the presented fifth-degree term in part (a).

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