2021

AP® CollegeBoard

AP® Calculus BC

Sample Student Responses and Scoring Commentary

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Free Response Question 4

- ☑ Scoring Guideline
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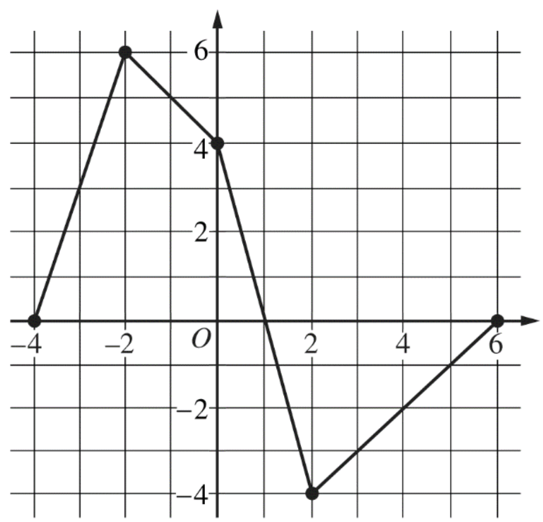
Part B (AB or BC): Graphing calculator not allowed
Question 4 — 9 points

General Scoring Notes

Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be correct to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.

Scoring guidelines and notes contain examples of the most common approaches seen in student responses. These guidelines can be applied to alternate approaches to ensure that these alternate approaches are scored appropriately.



Graph of f

Let f be a continuous function defined on the closed interval $-4 \le x \le 6$. The graph of f, consisting of four line segments, is shown above. Let G be the function defined by $G(x) = \int_0^x f(t)\,dt$.

Model Solution	Scoring	
$G'(x) = f(x)$ in any part of the response.	$G'(x) = f(x)$	**1 point**

Scoring notes:

- This "global point" can be earned in any one part. Expressions that show this connection and therefore earn this point include: $G' = f$, $G'(x) = f(x)$, $G''(x) = f'(x)$ in part (a), $G'(3) = f(3)$ in part (b), or $G'(2) = f(2)$ in part (c).

Total 1 point

(a) On what open intervals is the graph of G concave up? Give a reason for your answer.

$G'(x) = f(x)$ The graph of G is concave up for $-4 < x < -2$ and $2 < x < 6$, because $G' = f$ is increasing on these intervals.	Answer with reason	**1 point**

Scoring notes:

- Intervals may also include one or both endpoints.

Total for part (a) **1 point**

(b) Let P be the function defined by $P(x) = G(x) \cdot f(x)$. Find $P'(3)$.

$P'(x) = G(x) \cdot f'(x) + f(x) \cdot G'(x)$ $P'(3) = G(3) \cdot f'(3) + f(3) \cdot G'(3)$	Product rule	**1 point**
Substituting $G(3) = \int_0^3 f(t)\,dt = -3.5$ and $G'(3) = f(3) = -3$ into the above expression for $P'(3)$ gives the following:	$G(3)$ or $G'(3)$	**1 point**
$P'(3) = -3.5 \cdot 1 + (-3) \cdot (-3) = 5.5$	Answer	**1 point**

Scoring notes:

- The first point is earned for the correct application of the product rule in terms of x or in the evaluation of $P'(3)$. Once earned, this point cannot be lost.
- The second point is earned by correctly evaluating $G(3) = -3.5$, $G'(3) = -3$, or $f(3) = -3$.
- To be eligible to earn the third point a response must have earned the first two points.
- Simplification of the numerical value is not required to earn the third point.

Total for part (b) **3 points**

(c) Find $\lim_{x \to 2} \dfrac{G(x)}{x^2 - 2x}$.

$\lim_{x \to 2} \left(x^2 - 2x\right) = 0$ Because G is continuous for $-4 \le x \le 6$, $\lim_{x \to 2} G(x) = \int_0^2 f(t)\,dt = 0$. Therefore, the limit $\lim_{x \to 2} \dfrac{G(x)}{x^2 - 2x}$ is an indeterminate form of type $\dfrac{0}{0}$.	Uses L'Hospital's Rule	**1 point**
Using L'Hospital's Rule, $\lim_{x \to 2} \dfrac{G(x)}{x^2 - 2x} = \lim_{x \to 2} \dfrac{G'(x)}{2x - 2}$ $= \lim_{x \to 2} \dfrac{f(x)}{2x - 2} = \dfrac{f(2)}{2} = \dfrac{-4}{2} = -2$	Answer with justification	**1 point**

Scoring notes:

- To earn the first point the response must show $\lim_{x \to 2} \left(x^2 - 2x\right) = 0$ and $\lim_{x \to 2} G(x) = 0$ and must show a ratio of the two derivatives, $G'(x)$ and $2x - 2$. The ratio may be shown as evaluations of the derivatives at $x = 2$, such as $\dfrac{G'(2)}{2}$.
- To earn the second point the response must evaluate correctly with appropriate limit notation. In particular the response must include $\lim_{x \to 2} \dfrac{G'(x)}{2x - 2}$ or $\lim_{x \to 2} \dfrac{f(x)}{2x - 2}$.
- With any linkage errors (such as $\dfrac{G'(x)}{2x - 2} = \dfrac{f(2)}{2}$), the response does not earn the second point.

Total for part (c) 2 points

(d) Find the average rate of change of G on the interval $[-4, 2]$. Does the Mean Value Theorem guarantee a value c, $-4 < c < 2$, for which $G'(c)$ is equal to this average rate of change? Justify your answer.

$G(2) = \int_0^2 f(t)\,dt = 0$ and $G(-4) = \int_0^{-4} f(t)\,dt = -16$ Average rate of change $= \dfrac{G(2) - G(-4)}{2 - (-4)} = \dfrac{0 - (-16)}{6} = \dfrac{8}{3}$	Average rate of change	**1 point**
Yes, $G'(x) = f(x)$ so G is differentiable on $(-4, 2)$ and continuous on $[-4, 2]$. Therefore, the Mean Value Theorem applies and guarantees a value c, $-4 < c < 2$, such that $G'(c) = \dfrac{8}{3}$.	Answer with justification	**1 point**

Scoring notes:

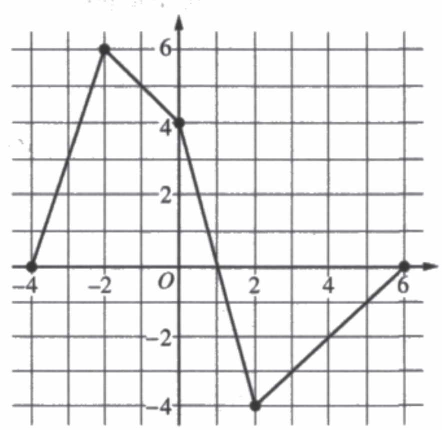
- To earn the first point a response must present at least a difference and a quotient and a correct evaluation. For example, $\dfrac{0+16}{6}$ or $\dfrac{G(2) - G(-4)}{6} = \dfrac{16}{6}$.
- Simplification of the numerical value is not required to earn the first point, but any simplification must be correct.
- The second point can be earned without the first point if the response has the correct setup but an incorrect or no evaluation of the average rate of change. The Mean Value Theorem need not be explicitly stated provided the conditions and conclusion are stated.

Total for part (d) **2 points**

Total for question 4 **9 points**

Sample 4A

Answer QUESTION 4 parts (a) and (b) on this page.



Graph of f

Response for question 4(a)

$G'(x) = f(x)$

$G''(x) = f'(x)$

On $(-4,-2)$ and $(2,6)$, $G(x)$ is concave up because $f(x)$ (which is eqival to $G'(x)$) has a positive slope / is increasing.

Response for question 4(b)

$P'(x) = G'(x)\,f(x) + f'(x)\,G(x)$

$P'(3) = G'(3)\,f(3) + f'(3)\,G(3)$

$G'(x) = f(x)$

$G'(3) = f(3) = -3$

$G(3) = \int_0^3 f(t)\,dt = -\frac{7}{2}$

$P'(3) = (-3)(-3) + (1)\left(-\frac{7}{2}\right)$

Answer QUESTION 4 parts (c) and (d) on this page.

Response for question 4(c)

$\lim_{x \to 2} G(x) = \lim_{x \to 2} (x^2 - 2x) = 0$ Must use l'hôpital's rule

$\llcorner \int_0^2 f(t)\,dt = 0$

$\lim_{x \to 2} \frac{G'(x)}{2x-2} = \frac{f(2)}{4-2} = \frac{-4}{2}$

Response for question 4(d)

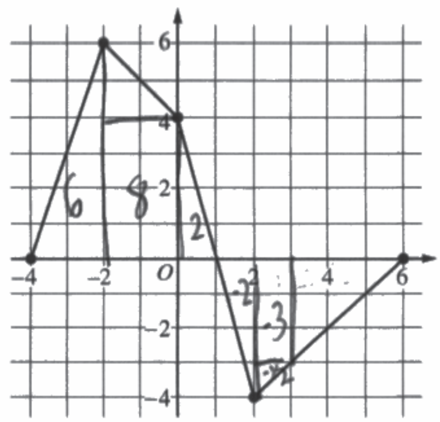
$\int_0^2 f(t)\,dt \qquad \int_0^{-4} f(t)\,dt = -(3+9+3+1) = -16$

$\text{AROC of } G = \frac{G(2) - G(-4)}{2-(-4)} = \frac{0-(-16)}{2+4} = \frac{16}{6} = \frac{8}{3}$

The meanvalue theorem does guarantee a value c, $-4 < c < 2$, for which $G'(c)$ is equal to average rate of change. This is because ~~$G'(x) = f(t)$~~ and ~~$x = t$~~ exists for all values, ~~$-4 < x = t < 2$~~, meaning that $G(x)$ is continuous on the closed interval and differentiable on the open interval.

4 4 4 4 4 NO CALCULATOR ALLOWED 4 4 4 4 4

Answer QUESTION 4 parts (a) and (b) on this page.



Graph of f

Response for question 4(a)

The graph of G is concave up on the intervals $(-4, -2) \cup (2, 6)$ because G is concave up when G' is increasing and $G'(x) = f(x)$.

Response for question 4(b)

$(4)(1)(\frac{1}{2}) = \cancel{2}\cancel{-2} - 3 - \frac{1}{2} = -3 - \frac{1}{2} = -\frac{6}{2} - \frac{1}{2} = -\frac{5}{2}$

$P(x) = G(x) \cdot f(x)$

$P'(x) = G'(x) \cdot f(x) + f'(x) \cdot G(x)$

$P'(3) = G'(3) \cdot f(3) + f'(3) \cdot G(3)$

$= (-3) \cdot (-3) + (-4) \cdot (-\frac{5}{2})$

$= 9 + 10 = 19$

$-\frac{5}{\cancel{2}_1} \cdot -\frac{\cancel{4}^2}{1} = 10$

$P'(3) \approx 19$

4 4 4 4 4 NO CALCULATOR ALLOWED 4 4 4 4 4

Answer QUESTION 4 parts (c) and (d) on this page.

Response for question 4(c)

$$\lim_{x\to 2} \frac{G(x) \to 0}{x^2-2x \to 0}$$ L'hopital rule

$$\lim_{x\to 2} \frac{G'(x)}{2x-2} = \frac{G'(2)}{2(2)-2} = \frac{-4}{2} = -2$$

Response for question 4(d)

$$\frac{1}{6}\int_{-4}^{2} -4 = -\frac{4}{6} = -\frac{2}{3}$$

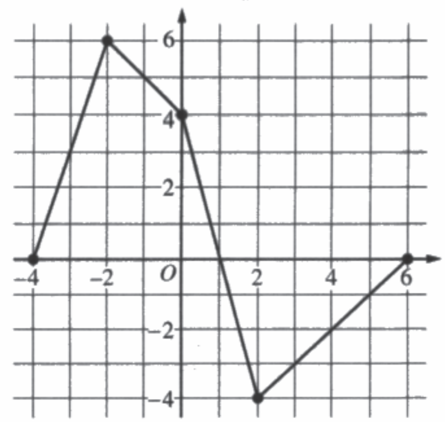
$$\frac{G(2)-G(-4)}{-4-2} = \frac{0-(-(6+8+2))}{-6} = \frac{16}{-6} = -\frac{8}{3}$$

The MVT doesn't guarantee the value of c, $-4 < c < 2$ for which $G'(c)$ is equal to the average rate of change of G because the values of the average rate of change are different since G' is the derivative G,

$$\frac{G'(2)-G'(-4)}{-4-2} = \frac{-4-0}{-6} = \frac{-4}{-6} = \frac{2}{3}$$

$$\frac{1}{2-4}\int_{-4}^{2} G(x)\,dx = \frac{1}{6}\cdot 16 = \frac{16}{6} = \frac{8}{3}$$

Answer QUESTION 4 parts (a) and (b) on this page.



Graph of f

Response for question 4(a)

$G(x) = \int_0^x f(t)\,dt$

$G(x) = \int f(x)\,dx$

$f(x) = G'(x)$

Because $f(x)$ is the antiderivative of $G(x)$ that means that $f(x) = G'(x)$. Therefore when the graph $f(x)$ = increasing, then that means $G(x)$ = concave up. G is concave up on the intervals $(-4, -2) \cup (2, 6)$ b/c f is increasing.

Response for question 4(b)

$p(x) = G(x) \cdot f(x)$

$p'(x) = G'(x) \cdot f(x) + f'(x) \cdot G(x)$

$p'(3) = G'(3) \cdot f(3) + f'(3) \cdot G(3)$

$p'(3) = -3 \cdot -3 + -3 \cdot -3$

$p'(3) = 9 + 9$

$p'(3) = 18$

Sample 4C

4 4 4 4 4 NO CALCULATOR ALLOWED 4 4 4 4 4

Answer QUESTION 4 parts (c) and (d) on this page.

Response for question 4(c)

$$\lim_{x \to 2} \frac{G(x)}{x^2 - 2x} \xrightarrow{\text{L'HOP}} \frac{G'(x)}{2x-2} = \frac{f(x)}{2x-2}$$

$G'(x) = f(x)$

$$= \frac{f(2)}{2(2)-2}$$

$$= \frac{-4}{2}$$

$$= -2$$

Response for question 4(d)

$$G(x) = \int_{-4}^{2} f(x)\,dx$$

$$G'(x) = f(x)\Big|_{-4}^{2}$$

$$= f(2) - f(-4)$$

$$= -4 - 0$$

$$= \boxed{-4}$$

$-4 < c < 2$ $G'(c) = -4$?

NO b/c c cannot equal -4, it can only be greater than it. Therefore the Mean Value Theorem does not guarantee a value.

Question 4

Note: Student samples are quoted verbatim and may contain spelling and grammatical errors.

Overview

In this problem the graph of a piecewise linear continuous function f for $-4 \le x \le 6$ is provided. It is also given that $G(x) = \int_0^x f(t)\,dt$.

In part (a) students were asked to provide the open intervals on which the graph of G is concave up. A correct response would use the Fundamental Theorem of Calculus to note that $G' = f$, and then report the two intervals where $G' = f$ is increasing.

In part (b) the function $P(x) = G(x) \cdot f(x)$ is defined and students were asked to find $P'(3)$. A correct response would use the product rule to find an expression for $P'(x)$, then use the graph of f to find numerical values of $f(3)$ and $f'(3)$, and use the Fundamental Theorem of Calculus to find $G(3)$ and $G'(3)$. The response would substitute these values into the expression for $P'(x)$ to provide the value of $P'(3)$.

In part (c) students were asked to find $\lim\limits_{x\to 2} \dfrac{G(x)}{x^2 - 2x}$. A correct response would use L'Hospital's Rule to find the limit after verifying that the limits of both the numerator and denominator are zero.

In part (d) students were asked to find the average rate of change of G on the interval $[-4, 2]$ and whether the Mean Value Theorem guarantees a value c, $-4 < c < 2$, with $G'(c)$ equal to this average rate of change. A correct response would determine the average rate of change as a difference quotient, $\dfrac{G(2) - G(-4)}{2 - (-4)}$, with values $G(2) = 0$ and $G(-4) = -16$ found as areas under the graph of f. The response should then conclude that the Mean Value Theorem does guarantee such a value of c because $G' = f$ is differentiable and, therefore, continuous on the given interval.

Sample: 4A
Score: 9

The response earned 9 points: 1 global point, 1 point in part (a), 3 points in part (b), 2 points in part (c), and 2 points in part (d). The global point was earned in the first line of part (a) with the statement $G'(x) = f(x)$. In part (a) the response earned the point with the presentation of the correct intervals and the reason "$f(x)$ which is equal to $G'(x)$ has a positive slope/is increasing." In part (b) the response earned the first point with the correct product rule presentation in the first line. The second point was earned for the correct values for both $G'(3)$ and $G(3)$ although only one correct value was necessary for this point. The third point was earned with the expression $(-3)(-3)+(1)\left(-\frac{7}{2}\right)$. Simplification of this expression is not necessary. In part (c) the response earned the first point with the extended equation of limits in the first line and the ratio of derivatives in the second line. The second point was earned with the correct ratio of derivatives accompanied by limit notation and the correct (unsimplified) answer. In part (d) the response earned the first point for a valid attempt to calculate the average rate of change of G and a correct result. The second point was earned with the answer, "The mean value theorem does guarantee a value c," and the statement that $G(x)$ is both differentiable and continuous.

Question 4 (continued)

Sample: 4B
Score: 6

The response earned 6 points: 1 global point, 1 point in part (a), 2 points in part (b), 2 points in part (c), and no points in part (d). The global point was earned in part (a) with the statement $G'(x) = f(x)$. In part (a) the response earned the point with the presentation of the correct intervals and the reason that G' is increasing. In part (b) the response earned the first point with the correct product rule in the second line. The second point was earned with the value of $G'(3)$ as -3 in the fourth line. Note that the incorrect value of $G(3)$ did not affect this point because only one correct value of $G(3)$ or $G'(3)$ is required. The third point was not earned due to the incorrect final answer. In part (c) the first point was earned with the arrows pointing from the numerator and denominator to the value 0 and by the ratio of derivatives. The second point was earned with the correct ratio of derivatives accompanied by limit notation and the correct answer. In part (d) the first point was not earned because the average rate of change presented is not correct (denominator should be $2-(-4)$). Because this is not a valid average rate of change form, the response is not eligible for the second point.

Sample: 4C
Score: 4

The response earned 4 points: 1 global point, 1 point in part (a), 2 points in part (b), no points in part (c), and no points in part (d). The global point was earned in the third line of part (a) with the statement $f(x) = G'(x)$. In part (a) the response earned the point with the presentation of the correct intervals and the reason "when $f(x) =$ increasing, then $G(x) =$ concave up." In part (b) the response earned the first point with the correct product rule in the second line. The second point was earned for the correct value for $G(3)$. The value for $G'(3)$ is incorrect, but only one correct value is necessary for this point. The third point was not earned because the final answer is incorrect. In part (c) the response did not earn the first point because there is no evidence of $\lim_{x\to 2} G(x) = 0$ or $\lim_{x\to 2}\left(x^2 - 2x\right) = 0$ given. The second point was not earned because the ratio of derivatives does not have limit notation. In part (d) the response did not earn the first point because there is not an attempt to calculate the average rate of change of G. The response is not eligible for the second point.