2021

AP® CollegeBoard

AP® Calculus BC

Sample Student Responses and Scoring Commentary

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Free Response Question 5

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Part B (BC): Graphing calculator not allowed

Question 5 — 9 points

General Scoring Notes

Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be correct to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.

Scoring guidelines and notes contain examples of the most common approaches seen in student responses. These guidelines can be applied to alternate approaches to ensure that these alternate approaches are scored appropriately.

Let $y = f(x)$ be the particular solution to the differential equation $\frac{dy}{dx} = y \cdot (x \ln x)$ with initial condition $f(1) = 4$. It can be shown that $f''(1) = 4$.

	Model Solution	Scoring	
(a)	Write the second-degree Taylor polynomial for f about $x = 1$. Use the Taylor polynomial to approximate $f(2)$.		
	$f'(1) = \left.\frac{dy}{dx}\right\|_{(x,\,y)=(1,\,4)} = 4 \cdot (1 \ln 1) = 0$ The second-degree Taylor polynomial for f about $x = 1$ is $f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 = 4 + 0(x-1) + \frac{4}{2}(x-1)^2$.	Polynomial	**1 point**
	$f(2) \approx 4 + 2(2-1)^2 = 6$	Approximation	**1 point**

Scoring notes:

- The first point is earned for $4 + \frac{4 \cdot \ln 1}{1!}(x-1)^1 + \frac{4}{2!}(x-1)^2$ or any correctly simplified equivalent expression. A term involving $(x-1)$ is not necessary. The polynomial must be written about (centered at) $x = 1$.
- If the first point is earned, the second point is earned for just "6" with no additional supporting work.
- If the polynomial is never explicitly written, the first point is not earned. In this case, to earn the second point supporting work of at least "$4 + 2(1)$" is required.

Total for part (a) 2 points

(b) Use Euler's method, starting at $x = 1$ with two steps of equal size, to approximate $f(2)$. Show the work that leads to your answer.

$f(1.5) \approx f(1) + 0.5 \cdot \left.\frac{dy}{dx}\right\|_{(x,\,y)=(1,\,4)} = 4 + 0.5 \cdot 0 = 4$ $f(2) \approx f(1.5) + 0.5 \cdot \left.\frac{dy}{dx}\right\|_{(x,\,y)=(1.5,\,4)}$	Euler's method with two steps	**1 point**
$\approx 4 + 0.5 \cdot 4 \cdot (1.5 \ln 1.5) = 4 + 3 \ln 1.5$	Answer	**1 point**

Scoring notes:

- The first point is earned for two steps (of size 0.5) of Euler's method, with at most one error. If there is any error, the second point is not earned.
- To earn the first point a response must contain two Euler steps, $\Delta x = 0.5$, use of the correct expression for $\frac{dy}{dx}$, and use of the initial condition $f(1) = 4$.
    - The two Euler steps may be explicit expressions or may be presented in a table. Here is a minimal example of a (correctly labeled) table.

x	y	$\Delta y = \frac{dy}{dx} \cdot \Delta x$ or $\Delta y = \frac{dy}{dx} \cdot (0.5)$
1	4	0
1.5	4	$3 \ln 1.5$
2	$4 + 3 \ln 1.5$	

    - Note: In the presence of the correct answer, such a table does not need to be labeled in order to earn both points. In the presence of an incorrect answer, the table must be correctly labeled for the response to earn the first point.
- A single error in computing the approximation of $f(1.5)$ is not considered a second error if that incorrect value is imported correctly into an approximation of $f(2)$.
- Both points are earned for "$4 + 0.5 \cdot 0 + 0.5 \cdot 4 \cdot (1.5 \ln 1.5)$" or "$4 + 0.5 \cdot 4 \cdot (1.5 \ln 1.5)$".
- Both points are earned for presenting the ordered pair $(2,\ 4 + 3 \ln 1.5)$ with sufficient supporting work.

Total for part (b) 2 points

(c) Find the particular solution $y = f(x)$ to the differential equation $\frac{dy}{dx} = y \cdot (x \ln x)$ with initial condition $f(1) = 4$.

$\frac{1}{y}\,dy = x \ln x\,dx$	Separation of variables	**1 point**
Using integration by parts, $u = \ln x \quad du = \frac{1}{x}\,dx$ $dv = x\,dx \quad v = \frac{x^2}{2}$ $\int x \ln x\,dx = \frac{x^2}{2} \cdot \ln x - \int \frac{x^2}{2} \cdot \frac{1}{x}\,dx = \frac{x^2 \ln x}{2} - \frac{x^2}{4} + C$	Antiderivative for $x \ln x$	**1 point**
$\ln\lvert y \rvert = \frac{x^2 \ln x}{2} - \frac{x^2}{4} + C$	Antiderivative for $\frac{1}{y}$	**1 point**
$\ln 4 = 0 - \frac{1}{4} + C \Rightarrow C = \ln 4 + \frac{1}{4}$	Constant of integration and uses initial condition	**1 point**
$y = e^{\left(\frac{x^2 \ln x}{2} - \frac{x^2}{4} + \ln 4 + \frac{1}{4}\right)}$ Note: This solution is valid for $x > 0$.	Solves for y	**1 point**

Scoring notes:

- A response with no separation of variables earns 0 out of 5 points. If an error in separation results in one side being correct ($\frac{1}{y}\,dy$ or $x \ln x\,dx$), the response is only eligible to earn the corresponding antiderivative point.
- The third point (antiderivative of $\frac{1}{y}$) can be earned for either $\ln y$ or $\ln\lvert y \rvert$.
- A response with no constant of integration can earn at most 3 out of 5 points.
- A response is eligible for the fourth point if it has earned the first point and at least 1 of the 2 antiderivative points.
- A response earns the fourth point by correctly including the constant of integration in an equation and then replacing x with 1 and y with 4.
- A response is eligible for the fifth point only if it has earned the first 4 points.

Total for part (c) **5 points**

Total for question 5 **9 points**

5 5 5 5 5 NO CALCULATOR ALLOWED 5 5 5 5 5

Answer QUESTION 5 parts (a) and (b) on this page.

Response for question 5(a)

$\frac{dy}{dx} = 4 \cdot \ln 1$

$P_2(x) = 4 + \cancel{\ln 1}(x-1) + \frac{4(x-1)^2}{2!}$ $P_2(x) = 4 + \frac{4(x-1)^2}{2!}$

$P_2(2) = 4 + \frac{4(2-1)^2}{2!} = 4 + 2 = \boxed{6}$

Response for question 5(b)

$\frac{dy}{dx} = 4 \cdot 1.5 \ln 1.5$
$6 \ln 1.5$

(x, y)	Δx	$\frac{dy}{dx}$	Δy
$(1, 4)$	$.5$	0	0
$(1.5, 4)$	$.5$	$6 \ln 1.5$	$3 \ln 1.5$
$(2, \ )$			

$\boxed{f(2) = 4 + 3 \ln 1.5}$

5 5 5 5 5 NO CALCULATOR ALLOWED 5 5 5 5 5

Answer QUESTION 5 part (c) on this page.

Response for question 5(c)

$\frac{dy}{dx} = y \cdot x\ln x$

$\frac{1}{y}dy = x\ln x\,dx$

$\int \frac{1}{y}dy = \int x\ln x\,dx$

$u = \ln x \qquad dv = x\,dx$

$du = \frac{1}{x}dx \qquad v = \frac{1}{2}x^2$

$\ln y = \frac{1}{2}x^2\ln x - \int \frac{1}{2}x\,dx$

$\ln y = \frac{1}{2}x^2\ln x - \frac{1}{4}x^2 + C$

$\ln 4 = \frac{1}{2}\ln 1 - \frac{1}{4} + C$

$\ln 4 + \frac{1}{4} = C$

$\ln y = \frac{1}{2}x^2\ln x - \frac{1}{4}x^2 + \ln 4 + \frac{1}{4}$

$\boxed{y = 4e^{\frac{1}{2}x^2\ln x - \frac{1}{4}x^2 + \frac{1}{4}}}$

5 5 5 5 5 NO CALCULATOR ALLOWED 5 5 5 5 5

Answer QUESTION 5 parts (a) and (b) on this page.

Response for question 5(a)

$f(1) + f'(1)(x-1) + \frac{f''(1)(x-1)^2}{2!}$

$T_2 = 4 + 0(x-1) + \frac{4(x-1)^2}{2!}$

$f(2) \approx 4 + 0(2-1) + \frac{4(2-1)^2}{2!}$

$= 4 + \frac{4}{2} = \boxed{4.5}$

Response for question 5(b)

x	y	Δx	$\frac{dy}{dx}$	$\Delta y = \Delta x\left(\frac{dy}{dx}\right)$	$(x+\Delta x, y+\Delta y)$
1	4	.5	0	0	$(1.5, 4)$
1.5	4	.5	$6\ln(1.5)$	$3\ln(1.5)$	$(2, 4+3\ln(1.5))$
2	$4+3\ln(1.5)$				

$f(2) \approx 4 + 3\ln(1.5)$

Answer QUESTION 5 part (c) on this page.

Response for question 5(c)

LIPET

$uv - \int v\,du$ $\quad u = \ln x \quad v = \frac{1}{2}x^2$

$du = \frac{1}{x}dx \quad dv = x\,dx$

$\frac{1}{2}x^2 \ln x - \frac{1}{2}\int x^2 \cdot \frac{1}{x}\,dx$

$= \frac{1}{2}x^2 \ln x - \frac{1}{2}\int x\,dx$

$= \frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 + C$

$\frac{dy}{dx} = y \cdot (x \ln x)$

$dy = y \cdot (x \ln x)\,dx$

$\int \frac{1}{y}\,dy = \int (x \ln x)\,dx$

$\ln|y| = \frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 + C \qquad f(1) = 4$

$\ln 4 = -\frac{1}{4} + C$

$C = \ln 4 + \frac{1}{4}$

$e^{\ln|y|} = e^{\frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 + \ln 4 + \frac{1}{4}}$

$y = e^{\frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 + \underbrace{\ln 4 + \frac{1}{4}}_{C}}$

$y = Ce^{kt} = (\ln 4 + \frac{1}{4})e^{\frac{1}{2}x^2 \ln x - \frac{1}{4}x^2}$

5 5 5 5 5 NO CALCULATOR ALLOWED 5 5 5 5 5

Answer QUESTION 5 parts (a) and (b) on this page.

Response for question 5(a)

$f(1) = 4$

$f'(1) = \frac{dy}{dx}\Big|_{(1,4)} = 4(1)\ln 1 = 0$

$f''(1) = \frac{d^2y}{dx^2}\Big|_{(1,4)} \Rightarrow \frac{d}{dx}[yx\ln x]\Big|_{(1,4)}$

$f \approx P_2(x) = 4 + \frac{f''(1)(x-1)^2}{2} = 4 + \frac{\frac{d}{dx}[yx\ln x]\big|_{(1,4)}(x-1)^2}{2}$

$f(2) \approx 4 + \frac{\frac{d}{dx}[yx\ln x]\big|_{(1,4)}(2-1)^2}{2} \rightarrow \boxed{f(2) \approx 4 + \frac{\frac{d}{dx}[yx\ln x]\big|_{(1,4)}}{2}}$

Response for question 5(b)

$f(1) = 4$

$f(1.5) = 4 + [4 \cdot 1(\cancel{\ln 1})](.5) = 6$

$\boxed{f(2) = 6 + [6 \cdot \tfrac{3}{2}\ln\tfrac{3}{2}](.5)}$

5 5 5 5 5 NO CALCULATOR ALLOWED 5 5 5 5 5

Answer QUESTION 5 part (c) on this page.

Response for question 5(c)

$\frac{dy}{dx} = yx\ln x$

$\int \frac{1}{y} dy = \int x \ln x \, dx$

integration by parts

$u = x, \quad dv = \ln x \, dx$

$du = 1\,dx \quad v = \frac{1}{x}$

$\int x \ln x \, dx = x\left(\frac{1}{x}\right) - \int \frac{1}{x}(1) dx$

$= 1 - \ln|x| + C$

$\ln|y| = 1 - \ln|x| + C$

find C, $f(1) = 4$

$\ln 4 = 1 - \cancel{\ln(1)} + C$

$C = \ln 4 - 1$

$\ln|y| = 1 - \ln|x| + \ln 4 - 1$

$\ln|y| = \ln 4 - \ln|x|$

$e^{\ln 4 - \ln|x|} = y$

Question 5

Note: Student samples are quoted verbatim and may contain spelling and grammatical errors.

Overview

In this problem $y = f(x)$ is the particular solution to $\frac{dy}{dx} = y \cdot (x \ln x)$ with $f(1) = 4$ and students were told that $f''(1) = 4.$ In part (a) students were asked to write the second-degree Taylor polynomial for f about $x = 1$ and to use the polynomial to approximate $f(2)$. A correct response would determine that $f'(1) = 0$ and use this value and the given values of $f(1)$ and $f''(1)$ to write the polynomial $4 + 2(x-1)^2$. The response would then find an approximation of $f(2) \approx 6.$

In part (b) students were asked to use Euler's method to approximate $f(2)$ using two steps of equal size starting at $x = 1.$ A correct response would use Euler's method with $\Delta x = 0.5$ to first approximate $f(1.5) \approx 4$ and then use that value with Euler's method to approximate $f(2) \approx 4 + 3\ln 1.5.$

In part (c) students were asked to find the particular solution $y = f(x)$ with initial condition $f(1) = 4.$ A correct response would use integration by parts to find $\ln|y| = \int x \ln x \, dx = \frac{x^2 \ln x}{2} - \frac{x^2}{4} + C,$ with $C = \ln 4 + \frac{1}{4}$ determined from the initial condition. Then solving for y results in the solution $y = e^{\left(\frac{x^2 \ln x}{2} - \frac{x^2}{4} + \ln 4 + \frac{1}{4}\right)}.$

Sample: 5A
Score: 9

The response earned 9 points: 2 points in part (a), 2 points in part (b), and 5 points in part (c). In part (a) the response earned the first point with a correct expression for the Taylor polynomial in line 2 on the left. Simplification is not necessary. The response earned the second point with a correct approximation of $f(2)$ in line 3 on the left. Again simplification is not necessary. In part (b) the response earned the first point in the table: two Euler steps are visible in the rows, $\Delta x = 0.5$ is given in the second column, the correct values for $\frac{dy}{dx}$ are given in the third column, and the initial condition $f(1) = 4$ is used in the first row. Note that because the final answer is correct in this case, the table does not need to be labeled. The response earned the second point with the correct answer boxed beneath the table. In part (c) the response earned the first point with a correct separation of variables in line 2. The response earned the second point with a correct antiderivative of $x \ln x$ in line 5 on the right. The response earned the third point with a correct antiderivative of $\frac{1}{y}$ in line 4 on the left. No absolute value signs are necessary. The response is eligible for the fourth point. The response earned the fourth point with the correct inclusion of the constant of integration in line 5 and the correct substitution of 1 for x and 4 for y in line 6. The response is eligible for the fifth point, which it earned with the correct answer in line 9.

Question 5 (continued)

Sample: 5B
Score: 7

The response earned 7 points: 1 point in part (a), 2 points in part (b), and 4 points in part (c). In part (a) the response earned the first point with a correct expression for the Taylor polynomial in line 2. Simplification is not necessary. The response did not earn the second point. While the correct approximation for $f(2)$ is shown in line 3 and in line 4 on the left, the boxed result is simplified incorrectly. In part (b) the response earned the first point: two Euler steps are seen in the second and third rows of the table, the step size $\Delta x = 0.5$ is shown in the third column of the table, the correct expression for $\frac{dy}{dx}$ is used to create the fourth column of the table, and the initial condition $f(1) = 4$ is shown in the first two entries of the second row of the table. Note that because the answer is correct in this case, the table does not need to be labeled. The response earned the second point with a correct answer below the table. In part (c) the response earned the first point with a correct separation of variables in line 3 on the left side of the page. The response earned the second point with a correct antiderivative of $x \ln x$ in lines 1-5 on the right side of the page. The response earned the third point with a correct antiderivative of $\frac{1}{y}$ in line 4 on the left side of the page. The response is eligible for the fourth point as it has earned the first point and at least one of the antiderivative points. The response earned the fourth point with a correct inclusion of the constant of integration in line 3 on the left side of the page and the substitution of 1 for x and 4 for y in line 4 on the left side of the page. The response did not earn the fifth point as the answer in line 9 on the left side of the page is incorrect. Note that the answer given in line 8 of the left side of the page is correct, but is simplified incorrectly.

Sample: 5C
Score: 4

The response earned 4 points: no points in part (a), 1 point in part (b), and 3 points in part (c). In part (a) the response did not earn the first point because no correct Taylor polynomial is presented. The response did not earn the second point because no correct approximation for $f(2)$ is presented. In part (b) the response earned the first point: two Euler steps are shown in lines 2 and 3, the step size $\Delta x = 0.5$ is used in lines 2 and 3, the correct expression for the derivative is evaluated in lines 2 and 3, and the initial condition $f(1) = 4$ is stated in line 1. Note that the response contains a single error: the value of the approximation for $f(1.5)$ in line 2 is simplified incorrectly. Importing this incorrect value correctly into line 3 is not an error. Because only one mistake is made, the response is still eligible for the first point, which it earned. The response did not earn the second point because the boxed answer is incorrect. In part (c) the response earned the first point with a correct separation of variables in line 2 on the left side of the page. The response did not earn the second point as the antiderivative for $x \ln x$ is incorrect. The response earned the third point with a correct antiderivative for $\frac{1}{y}$ in line 3 on the left side of the page. The response is eligible for the fourth point because it earned the first point and one of the two antiderivative points. The response earned the fourth point with a correct inclusion of a constant of integration in line 3 on the left side of the page and a correct substitution of 1 for x and 4 for y in line 5 on the left side of the page. The response is not eligible for the fifth point as it did not earn all of the first four points.