2021

AP® | CollegeBoard

AP® Calculus BC

Scoring Guidelines

Part A (AB or BC): Graphing calculator required

Question 1 — 9 points

General Scoring Notes

Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be correct to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.

Scoring guidelines and notes contain examples of the most common approaches seen in student responses. These guidelines can be applied to alternate approaches to ensure that these alternate approaches are scored appropriately.

r (centimeters)	0	1	2	2.5	4
$f(r)$ (milligrams per square centimeter)	1	2	6	10	18

The density of a bacteria population in a circular petri dish at a distance r centimeters from the center of the dish is given by an increasing, differentiable function f, where $f(r)$ is measured in milligrams per square centimeter. Values of $f(r)$ for selected values of r are given in the table above.

Model Solution	Scoring	
(a) Use the data in the table to estimate $f'(2.25)$. Using correct units, interpret the meaning of your answer in the context of this problem.		
$f'(2.25) \approx \dfrac{f(2.5)-f(2)}{2.5-2} = \dfrac{10-6}{0.5} = 8$	Estimate	**1 point**
At a distance of $r = 2.25$ centimeters from the center of the petri dish, the density of the bacteria population is increasing at a rate of 8 milligrams per square centimeter per centimeter.	Interpretation with units	**1 point**

Scoring notes:

- To earn the first point the response must provide both a difference and a quotient and must explicitly use values of f from the table.
- Simplification of the numerical value is not required to earn the first point. If the numerical value is simplified, it must be correct.
- The interpretation requires all of the following: distance $r = 2.25$, density of bacteria (population) is increasing or changing, at a rate of 8, and units of milligrams per square centimeter per centimeter.
- The second point (interpretation) cannot be earned without a nonzero presented value for $f'(2.25)$.
- To earn the second point the interpretation must be consistent with the presented nonzero value for $f'(2.25)$. In particular, if the presented value for $f'(2.25)$ is negative, the interpretation must include "decreasing at a rate of $|f'(2.25)|$" or "changing at a rate of $f'(2.25)$." The second point cannot be earned for an incorrect statement such as "the bacteria density is decreasing at a rate of -8 … " even for a presented $f'(2.25) = -8$.
- The units ($\text{mg/cm}^2/\text{cm}$) may be attached to the estimate of $f'(2.25)$ and, if so, do not need to be repeated in the interpretation.
- If units attached to the estimate do not agree with units in the interpretation, read the units in the interpretation.

Total for part (a) 2 points

(b) The total mass, in milligrams, of bacteria in the petri dish is given by the integral expression $2\pi\int_0^4 r\,f(r)\,dr$. Approximate the value of $2\pi\int_0^4 r\,f(r)\,dr$ using a right Riemann sum with the four subintervals indicated by the data in the table.

$2\pi\int_0^4 r\,f(r)\,dr \approx 2\pi(1\cdot f(1)\cdot(1-0) + 2\cdot f(2)\cdot(2-1)$ $+ 2.5\cdot f(2.5)\cdot(2.5-2) + 4\cdot f(4)\cdot(4-2.5))$	Right Riemann sum setup	**1 point**
$= 2\pi(1\cdot 2\cdot 1 + 2\cdot 6\cdot 1 + 2.5\cdot 10\cdot 0.5 + 4\cdot 18\cdot 1.5)$ $= 269\pi = 845.088$	Approximation	**1 point**

Scoring notes:

- The presence or absence of 2π has no bearing on earning the first point.
- The first point is earned for a sum of four products with at most one error in any single value among the four products. Multiplication by 1 in any term does not need to be shown, but all other products must be explicitly shown.
- A response of $1\cdot f(1)\cdot(1-0) + 2\cdot f(2)\cdot(2-1) + 2.5\cdot f(2.5)\cdot(2.5-2) + 4\cdot f(4)\cdot(4-2.5)$ earns the first point but not the second.
- A response with any error in the Riemann sum is not eligible for the second point.
- A response that provides a completely correct left Riemann sum for $2\pi\int_0^4 r\,f(r)\,dr$ and approximation (91π) earns one of the two points. A response that has any error in a left Riemann sum or evaluation for $2\pi\int_0^4 r\,f(r)\,dr$ earns no points.
- A response that provides a completely correct right Riemann sum for $2\pi\int_0^4 f(r)\,dr$ and approximation (80π) earns one of the two points. A response that has any error in a right Riemann sum or evaluation for $2\pi\int_0^4 f(r)\,dr$ earns no points.
- Simplification of the numerical value is not required to earn the second point. If a numerical value is given, it must be correct to three decimal places.

Total for part (b) 2 points

(c) Is the approximation found in part (b) an overestimate or underestimate of the total mass of bacteria in the petri dish? Explain your reasoning.

$\frac{d}{dr}(r\,f(r)) = f(r) + r\,f'(r)$	Product rule expression for $\frac{d}{dr}(r\,f(r))$	**1 point**
Because f is nonnegative and increasing, $\frac{d}{dr}(r\,f(r)) > 0$ on the interval $0 \le r \le 4.$ Thus, the integrand $r\,f(r)$ is strictly increasing. Therefore, the right Riemann sum approximation of $2\pi\int_0^4 r\,f(r)\,dr$ is an overestimate.	Answer with explanation	**1 point**

Scoring notes:

- To earn the second point a response must explain that $r\,f(r)$ is increasing and, therefore, the right Riemann sum is an overestimate. The second point can be earned without having earned the first point.
- A response that attempts to explain based on a left Riemann sum for $2\pi\int_0^4 r\,f(r)\,dr$ from part (b) earns no points.
- A response that attempts to explain based on a right Riemann sum for $2\pi\int_0^4 f(r)\,dr$ from part (b) earns no points.

Total for part (c) 2 points

(d) The density of bacteria in the petri dish, for $1 \le r \le 4$, is modeled by the function g defined by $g(r) = 2 - 16\left(\cos\left(1.57\sqrt{r}\right)\right)^3$. For what value of k, $1 < k < 4$, is $g(k)$ equal to the average value of $g(r)$ on the interval $1 \le r \le 4$?

Average value $= g_{\text{avg}} = \dfrac{1}{4-1}\displaystyle\int_1^4 g(r)\,dr$	Definite integral	**1 point**
$\dfrac{1}{4-1}\displaystyle\int_1^4 g(r)\,dr = 9.875795$	Average value	**1 point**
$g(k) = g_{\text{avg}} \Rightarrow k = 2.497$	Answer	**1 point**

Scoring notes:

- The first point is earned for a definite integral, with or without $\frac{1}{4-1}$ or $\frac{1}{3}$.
- A response that presents a definite integral with incorrect limits but a correct integrand earns the first point.
- Presentation of the numerical value 9.875795 is not required to earn the second point. This point can be earned by the average value setup: $\frac{1}{3}\int_1^4 g(r)\,dr$.
- Once earned for the average value setup, the second point cannot be lost. Subsequent errors will result in not earning the third point.
- The third point is earned only for the value $k = 2.497$.
- The third point cannot be earned without the second.
- Special case: A response that does not provide the average value setup but presents an average value of -13.955 is using degree mode on their calculator. This response would not earn the second point but could earn the third point for an answer of $k = 2.5$ (or 2.499).

Total for part (d) 3 points

Total for question 1 9 points

Part A (BC): Graphing calculator required

Question 2 — 9 points

General Scoring Notes

Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be correct to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.

Scoring guidelines and notes contain examples of the most common approaches seen in student responses. These guidelines can be applied to alternate approaches to ensure that these alternate approaches are scored appropriately.

For time $t \geq 0,$ a particle moves in the xy-plane with position $(x(t), y(t))$ and velocity vector $\left\langle (t-1)e^{t^2}, \sin\left(t^{1.25}\right) \right\rangle.$ At time $t = 0,$ the position of the particle is $(-2, 5).$

Model Solution	Scoring	
(a) Find the speed of the particle at time $t = 1.2.$ Find the acceleration vector of the particle at time $t = 1.2.$		
$\sqrt{(x'(1.2))^2 + (y'(1.2))^2} = 1.271488$ At time $t = 1.2,$ the speed of the particle is $1.271.$	Speed	**1 point**
$\langle x''(1.2), y''(1.2) \rangle = \langle 6.246630, 0.405125 \rangle$ At time $t = 1.2,$ the acceleration vector of the particle is $\langle 6.247 \text{ (or } 6.246), 0.405 \rangle.$	Acceleration vector	**1 point**

Scoring notes:

- Unsupported answers do not earn any points in this part.
- The acceleration vector may be presented with other symbols, for example $(\, , \,)$ or $[\, , \,],$ or the coordinates may be listed separately, as long as they are labeled.
- Degree mode: A response that presents answers obtained by using a calculator in degree mode does not earn the first point it would have otherwise earned. The response is generally eligible for all subsequent points (unless no answer is possible in degree mode or the question is made simpler by using degree mode). In degree mode, speed $= 0.844$ and $y''(1.2) = 0.023$ (or 0.022).

Total for part (a) — 2 points

(b) Find the total distance traveled by the particle over the time interval $0 \le t \le 1.2$.

$\int_0^{1.2} \sqrt{(x'(t))^2 + (y'(t))^2}\, dt = 1.009817$	Integrand	**1 point**
The total distance traveled by the particle over the time interval $0 \le t \le 1.2$ is 1.010 (or 1.009).	Answer	**1 point**

Scoring notes:

- The first point is earned by presenting the integrand $\sqrt{(x'(t))^2 + (y'(t))^2}$ in a definite integral with any limits. A definite integral with incorrect limits is not eligible for the second point.
- Once earned, the first point cannot be lost. Even in the presence of subsequent copy errors, the correct answer will earn the second point.
- If the first point is not earned because of a copy error, the second point is still earned for a correct answer.
- Unsupported answers will not earn either point.
- Degree mode: distance $= 0.677$ (or 0.676). (See degree mode statement in part (a).)

Total for part (b) 2 points

(c) Find the coordinates of the point at which the particle is farthest to the left for $t \geq 0$. Explain why there is no point at which the particle is farthest to the right for $t \geq 0$.

$x'(t) = (t-1)e^{t^2} = 0 \Rightarrow t = 1$	Sets $x'(t) = 0$	**1 point**
Because $x'(t) < 0$ for $0 < t < 1$ and $x'(t) > 0$ for $t > 1$, the particle is farthest to the left at time $t = 1$.	Explains leftmost position at $t = 1$	**1 point**
$x(1) = -2 + \int_0^1 x'(t)\,dt = -2.603511$ $y(1) = 5 + \int_0^1 y'(t)\,dt = 5.410486$	One coordinate of leftmost position	**1 point**
The particle is farthest to the left at point $(-2.604 \text{ (or } -2.603), 5.410)$.	Leftmost position	**1 point**
Because $x'(t) > 0$ for $t > 1$, the particle moves to the right for $t > 1$. Also, $x(2) = -2 + \int_0^2 x'(t)\,dt > -2 = x(0)$, so the particle's motion extends to the right of its initial position after time $t = 1$. Therefore, there is no point at which the particle is farthest to the right.	Explanation	**1 point**

Scoring notes:

- The second point is earned for presenting a valid reason why the particle is at its leftmost position at time $t = 1$. For example, a response could present the argument shown in the model solution, or it could indicate that the only critical point of $x(t)$ occurs at $t = 1$ and $x'(t)$ changes from negative to positive at this time.
- Unsupported positions $x(1)$ and/or $y(1)$ do not earn the third (or fourth) point(s).
- Writing $x(1) = \int_0^1 x'(t) - 2 = -2.603511$ does not earn the third (or fourth) point, because the missing dt makes this statement unclear or false. However, $x(1) = -2 + \int_0^1 x'(t) = -2.603511$ does earn the third point, because it is not ambiguous. Similarly, for $y(1)$.
- For the fourth point the coordinates of the leftmost point do not have to be written as an ordered pair as long as they are labeled as the x- and y-coordinates.
- To earn the last point a response must verify that the particle moves to the right of its initial position (as well as moves to the right for all $t > 1$). Note that there are several ways to demonstrate this.
- Degree mode: y-coordinate $= 5.008$ (or 5.007). (See degree mode statement in part (a).)

Total for part (c) **5 points**

Total for question 2 **9 points**

Part B (AB or BC): Graphing calculator not allowed
Question 3 — 9 points

General Scoring Notes

Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be correct to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.

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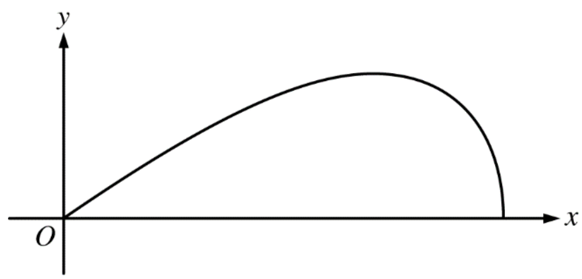

A company designs spinning toys using the family of functions $y = cx\sqrt{4 - x^2}$, where c is a positive constant. The figure above shows the region in the first quadrant bounded by the x-axis and the graph of $y = cx\sqrt{4 - x^2}$, for some c. Each spinning toy is in the shape of the solid generated when such a region is revolved about the x-axis. Both x and y are measured in inches.

Model Solution	Scoring	

(a) Find the area of the region in the first quadrant bounded by the x-axis and the graph of $y = cx\sqrt{4 - x^2}$ for $c = 6$.

$6x\sqrt{4 - x^2} = 0 \Rightarrow x = 0, x = 2$ $\text{Area} = \int_0^2 6x\sqrt{4 - x^2}\,dx$	Integrand	**1 point**
Let $u = 4 - x^2$. $du = -2x\,dx \Rightarrow -\frac{1}{2}\,du = x\,dx$ $x = 0 \Rightarrow u = 4 - 0^2 = 4$ $x = 2 \Rightarrow u = 4 - 2^2 = 0$ $\int_0^2 6x\sqrt{4 - x^2}\,dx = \int_4^0 6\left(-\frac{1}{2}\right)\sqrt{u}\,du = -3\int_4^0 u^{1/2}\,du = 3\int_0^4 u^{1/2}\,du$ $= 2u^{3/2}\Big\vert_{u=0}^{u=4} = 2 \cdot 8 = 16$	Antiderivative	**1 point**
The area of the region is 16 square inches.	Answer	**1 point**

Scoring notes:

- Units are not required for any points in this question and are not read if presented (correctly or incorrectly) in any part of the response.
- The first point is earned for presenting $cx\sqrt{4 - x^2}$ or $6x\sqrt{4 - x^2}$ as the integrand in a definite integral. Limits of integration (numeric or alphanumeric) must be presented (as part of the definite integral) but do not need to be correct in order to earn the first point.
- If an indefinite integral is presented with an integrand of the correct form, the first point can be earned if the antiderivative (correct or incorrect) is eventually evaluated using the correct limits of integration.
- The second point can be earned without the first point. The second point is earned for the presentation of a correct antiderivative of a function of the form $Ax\sqrt{4 - x^2}$, for any nonzero constant A. If the response has subsequent errors in simplification of the antiderivative or sign errors, the response will earn the second point but will not earn the third point.
- Responses that use u-substitution and have incorrect limits of integration or do not change the limits of integration from x- to u-values are eligible for the second point.
- The response is eligible for the third point only if it has earned the second point.
- The third point is earned only for the answer 16 or equivalent. In the case where a response only presents an indefinite integral, the use of the correct limits of integration to evaluate the antiderivative must be shown to earn the third point.
- The response cannot correct -16 to $+16$ in order to earn the third point; there is no possible reversal here.

Total for part (a) 3 points

(b) It is known that, for $y = cx\sqrt{4 - x^2}$, $\dfrac{dy}{dx} = \dfrac{c(4 - 2x^2)}{\sqrt{4 - x^2}}$. For a particular spinning toy, the radius of the largest cross-sectional circular slice is 1.2 inches. What is the value of c for this spinning toy?

The cross-sectional circular slice with the largest radius occurs where $cx\sqrt{4 - x^2}$ has its maximum on the interval $0 < x < 2$. $\dfrac{dy}{dx} = \dfrac{c(4 - 2x^2)}{\sqrt{4 - x^2}} = 0 \Rightarrow x = \sqrt{2}$	Sets $\dfrac{dy}{dx} = 0$	**1 point**
$x = \sqrt{2} \Rightarrow y = c\sqrt{2}\sqrt{4 - (\sqrt{2})^2} = 2c$ $2c = 1.2 \Rightarrow c = 0.6$	Answer	**1 point**

Scoring notes:

- The first point is earned for setting $\dfrac{dy}{dx} = 0$, $\dfrac{c(4 - 2x^2)}{\sqrt{4 - x^2}} = 0$, or $c(4 - 2x^2) = 0$.
- An unsupported $x = \sqrt{2}$ does not earn the first point.
- The second point can be earned without the first point but is earned only for the answer $c = 0.6$ with supporting work.

Total for part (b) 2 points

(c) For another spinning toy, the volume is 2π cubic inches. What is the value of c for this spinning toy?

$\text{Volume} = \int_0^2 \pi\left(cx\sqrt{4-x^2}\right)^2\,dx = \pi c^2\int_0^2 x^2\left(4-x^2\right)dx$	Form of the integrand Limits and constant	**1 point** **1 point**
$= \pi c^2\int_0^2\left(4x^2-x^4\right)dx = \pi c^2\left(\frac{4}{3}x^3-\frac{1}{5}x^5\Big\|_0^2\right)$	Antiderivative	**1 point**
$= \pi c^2\left(\frac{32}{3}-\frac{32}{5}\right) = \frac{64\pi c^2}{15}$ $\frac{64\pi c^2}{15} = 2\pi \Rightarrow c^2 = \frac{15}{32} \Rightarrow c = \sqrt{\frac{15}{32}}$	Answer	**1 point**

Scoring notes:

- The first point is earned for presenting an integrand of the form $A\left(x\sqrt{4-x^2}\right)^2$ in a definite integral with any limits of integration (numeric or alphanumeric) and any nonzero constant A. Mishandling the c will result in the response being ineligible for the fourth point.
- The second point can be earned without the first point. The second point is earned for the limits of integration, $x = 0$ and $x = 2$, and the constant π (but not for 2π) as part of an integral with a correct or incorrect integrand.
- If an indefinite integral is presented with the correct constant π, the second point can be earned if the antiderivative (correct or incorrect) is evaluated using the correct limits of integration.
- A response that presents $2 = \int_0^2\left(cx\sqrt{4-x^2}\right)^2 dx$ earns the first and second points.
- The third point is earned for presenting a correct antiderivative of the presented integrand of the form $A\left(x\sqrt{4-x^2}\right)^2$ for any nonzero A. If there are subsequent errors in simplification of the antiderivative, linkage errors, or sign errors, the response will not earn the fourth point.
- The fourth point cannot be earned without the third point. The fourth point is earned only for the correct answer. The expression does not need to be simplified to earn the fourth point.

Total for part (c) 4 points

Total for question 3 9 points

Part B (AB or BC): Graphing calculator not allowed
Question 4 — 9 points

General Scoring Notes

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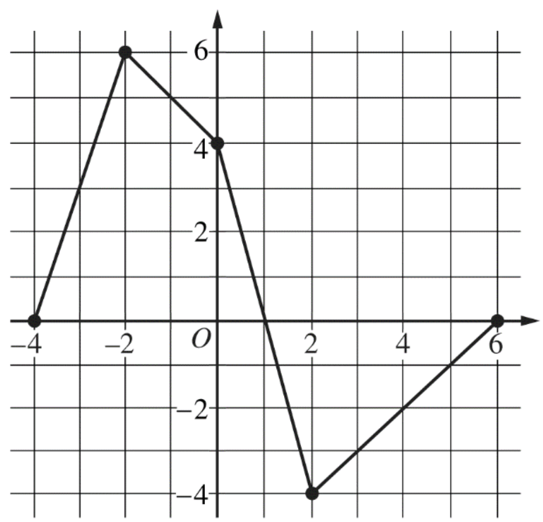

Graph of f

Let f be a continuous function defined on the closed interval $-4 \le x \le 6$. The graph of f, consisting of four line segments, is shown above. Let G be the function defined by $G(x) = \int_0^x f(t)\,dt$.

Model Solution	Scoring	
$G'(x) = f(x)$ in any part of the response.	$G'(x) = f(x)$	**1 point**

Scoring notes:

- This "global point" can be earned in any one part. Expressions that show this connection and therefore earn this point include: $G' = f$, $G'(x) = f(x)$, $G''(x) = f'(x)$ in part (a), $G'(3) = f(3)$ in part (b), or $G'(2) = f(2)$ in part (c).

Total **1 point**

(a) On what open intervals is the graph of G concave up? Give a reason for your answer.

$G'(x) = f(x)$ The graph of G is concave up for $-4 < x < -2$ and $2 < x < 6$, because $G' = f$ is increasing on these intervals.	Answer with reason	**1 point**

Scoring notes:

- Intervals may also include one or both endpoints.

Total for part (a) **1 point**

(b) Let P be the function defined by $P(x) = G(x) \cdot f(x)$. Find $P'(3)$.

$P'(x) = G(x) \cdot f'(x) + f(x) \cdot G'(x)$ $P'(3) = G(3) \cdot f'(3) + f(3) \cdot G'(3)$	Product rule	**1 point**
Substituting $G(3) = \int_0^3 f(t)\,dt = -3.5$ and $G'(3) = f(3) = -3$ into the above expression for $P'(3)$ gives the following:	$G(3)$ or $G'(3)$	**1 point**
$P'(3) = -3.5 \cdot 1 + (-3) \cdot (-3) = 5.5$	Answer	**1 point**

Scoring notes:

- The first point is earned for the correct application of the product rule in terms of x or in the evaluation of $P'(3)$. Once earned, this point cannot be lost.
- The second point is earned by correctly evaluating $G(3) = -3.5$, $G'(3) = -3$, or $f(3) = -3$.
- To be eligible to earn the third point a response must have earned the first two points.
- Simplification of the numerical value is not required to earn the third point.

Total for part (b) **3 points**

(c) Find $\lim_{x \to 2} \frac{G(x)}{x^2 - 2x}$.

$\lim_{x \to 2} \left(x^2 - 2x\right) = 0$ Because G is continuous for $-4 \le x \le 6$, $\lim_{x \to 2} G(x) = \int_0^2 f(t)\,dt = 0$. Therefore, the limit $\lim_{x \to 2} \frac{G(x)}{x^2 - 2x}$ is an indeterminate form of type $\frac{0}{0}$.	Uses L'Hospital's Rule	**1 point**
Using L'Hospital's Rule, $\lim_{x \to 2} \frac{G(x)}{x^2 - 2x} = \lim_{x \to 2} \frac{G'(x)}{2x - 2}$ $= \lim_{x \to 2} \frac{f(x)}{2x - 2} = \frac{f(2)}{2} = \frac{-4}{2} = -2$	Answer with justification	**1 point**

Scoring notes:

- To earn the first point the response must show $\lim_{x \to 2} \left(x^2 - 2x\right) = 0$ and $\lim_{x \to 2} G(x) = 0$ and must show a ratio of the two derivatives, $G'(x)$ and $2x - 2$. The ratio may be shown as evaluations of the derivatives at $x = 2$, such as $\frac{G'(2)}{2}$.
- To earn the second point the response must evaluate correctly with appropriate limit notation. In particular the response must include $\lim_{x \to 2} \frac{G'(x)}{2x - 2}$ or $\lim_{x \to 2} \frac{f(x)}{2x - 2}$.
- With any linkage errors (such as $\frac{G'(x)}{2x - 2} = \frac{f(2)}{2}$), the response does not earn the second point.

Total for part (c) 2 points

(d) Find the average rate of change of G on the interval $[-4, 2]$. Does the Mean Value Theorem guarantee a value c, $-4 < c < 2$, for which $G'(c)$ is equal to this average rate of change? Justify your answer.

$G(2) = \int_0^2 f(t)\,dt = 0$ and $G(-4) = \int_0^{-4} f(t)\,dt = -16$ Average rate of change $= \dfrac{G(2) - G(-4)}{2 - (-4)} = \dfrac{0 - (-16)}{6} = \dfrac{8}{3}$	Average rate of change	**1 point**
Yes, $G'(x) = f(x)$ so G is differentiable on $(-4, 2)$ and continuous on $[-4, 2]$. Therefore, the Mean Value Theorem applies and guarantees a value c, $-4 < c < 2$, such that $G'(c) = \frac{8}{3}$.	Answer with justification	**1 point**

Scoring notes:

- To earn the first point a response must present at least a difference and a quotient and a correct evaluation. For example, $\frac{0+16}{6}$ or $\frac{G(2) - G(-4)}{6} = \frac{16}{6}$.
- Simplification of the numerical value is not required to earn the first point, but any simplification must be correct.
- The second point can be earned without the first point if the response has the correct setup but an incorrect or no evaluation of the average rate of change. The Mean Value Theorem need not be explicitly stated provided the conditions and conclusion are stated.

Total for part (d) **2 points**

Total for question 4 **9 points**

Part B (BC): Graphing calculator not allowed
Question 5 — 9 points

General Scoring Notes
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Let $y = f(x)$ be the particular solution to the differential equation $\frac{dy}{dx} = y \cdot (x \ln x)$ with initial condition $f(1) = 4$. It can be shown that $f''(1) = 4$.

Model Solution	Scoring	
(a) Write the second-degree Taylor polynomial for f about $x = 1$. Use the Taylor polynomial to approximate $f(2)$.		
$f'(1) = \left.\frac{dy}{dx}\right\|_{(x,\,y)=(1,\,4)} = 4 \cdot (1 \ln 1) = 0$ The second-degree Taylor polynomial for f about $x = 1$ is $f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 = 4 + 0(x-1) + \frac{4}{2}(x-1)^2$.	Polynomial	**1 point**
$f(2) \approx 4 + 2(2-1)^2 = 6$	Approximation	**1 point**

Scoring notes:

- The first point is earned for $4 + \frac{4 \cdot \ln 1}{1!}(x-1)^1 + \frac{4}{2!}(x-1)^2$ or any correctly simplified equivalent expression. A term involving $(x-1)$ is not necessary. The polynomial must be written about (centered at) $x = 1$.
- If the first point is earned, the second point is earned for just "6" with no additional supporting work.
- If the polynomial is never explicitly written, the first point is not earned. In this case, to earn the second point supporting work of at least "$4 + 2(1)$" is required.

Total for part (a) — 2 points

(b) Use Euler's method, starting at $x = 1$ with two steps of equal size, to approximate $f(2)$. Show the work that leads to your answer.

$f(1.5) \approx f(1) + 0.5 \cdot \left.\frac{dy}{dx}\right\|_{(x,\,y)=(1,\,4)} = 4 + 0.5 \cdot 0 = 4$ $f(2) \approx f(1.5) + 0.5 \cdot \left.\frac{dy}{dx}\right\|_{(x,\,y)=(1.5,\,4)}$	Euler's method with two steps	**1 point**
$\approx 4 + 0.5 \cdot 4 \cdot (1.5 \ln 1.5) = 4 + 3 \ln 1.5$	Answer	**1 point**

Scoring notes:

- The first point is earned for two steps (of size 0.5) of Euler's method, with at most one error. If there is any error, the second point is not earned.
- To earn the first point a response must contain two Euler steps, $\Delta x = 0.5$, use of the correct expression for $\frac{dy}{dx}$, and use of the initial condition $f(1) = 4$.
  - The two Euler steps may be explicit expressions or may be presented in a table. Here is a minimal example of a (correctly labeled) table.

x	y	$\Delta y = \frac{dy}{dx} \cdot \Delta x$ or $\Delta y = \frac{dy}{dx} \cdot (0.5)$
1	4	0
1.5	4	$3 \ln 1.5$
2	$4 + 3 \ln 1.5$	

  - Note: In the presence of the correct answer, such a table does not need to be labeled in order to earn both points. In the presence of an incorrect answer, the table must be correctly labeled for the response to earn the first point.
- A single error in computing the approximation of $f(1.5)$ is not considered a second error if that incorrect value is imported correctly into an approximation of $f(2)$.
- Both points are earned for "$4 + 0.5 \cdot 0 + 0.5 \cdot 4 \cdot (1.5 \ln 1.5)$" or "$4 + 0.5 \cdot 4 \cdot (1.5 \ln 1.5)$".
- Both points are earned for presenting the ordered pair $(2, 4 + 3 \ln 1.5)$ with sufficient supporting work.

Total for part (b) **2 points**

(c) Find the particular solution $y = f(x)$ to the differential equation $\frac{dy}{dx} = y \cdot (x \ln x)$ with initial condition $f(1) = 4$.

$\frac{1}{y}\,dy = x \ln x\,dx$	Separation of variables	**1 point**
Using integration by parts, $u = \ln x \quad du = \frac{1}{x}\,dx$ $dv = x\,dx \quad v = \frac{x^2}{2}$ $\int x \ln x\,dx = \frac{x^2}{2} \cdot \ln x - \int \frac{x^2}{2} \cdot \frac{1}{x}\,dx = \frac{x^2 \ln x}{2} - \frac{x^2}{4} + C$	Antiderivative for $x \ln x$	**1 point**
$\ln\lvert y\rvert = \frac{x^2 \ln x}{2} - \frac{x^2}{4} + C$	Antiderivative for $\frac{1}{y}$	**1 point**
$\ln 4 = 0 - \frac{1}{4} + C \Rightarrow C = \ln 4 + \frac{1}{4}$	Constant of integration and uses initial condition	**1 point**
$y = e^{\left(\frac{x^2 \ln x}{2} - \frac{x^2}{4} + \ln 4 + \frac{1}{4}\right)}$ Note: This solution is valid for $x > 0$.	Solves for y	**1 point**

Scoring notes:

- A response with no separation of variables earns 0 out of 5 points. If an error in separation results in one side being correct ($\frac{1}{y}\,dy$ or $x \ln x\,dx$), the response is only eligible to earn the corresponding antiderivative point.
- The third point (antiderivative of $\frac{1}{y}$) can be earned for either $\ln y$ or $\ln\lvert y\rvert$.
- A response with no constant of integration can earn at most 3 out of 5 points.
- A response is eligible for the fourth point if it has earned the first point and at least 1 of the 2 antiderivative points.
- A response earns the fourth point by correctly including the constant of integration in an equation and then replacing x with 1 and y with 4.
- A response is eligible for the fifth point only if it has earned the first 4 points.

Total for part (c) **5 points**

Total for question 5 **9 points**

Part B (BC): Graphing calculator not allowed
Question 6 — 9 points

General Scoring Notes
Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be correct to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.

Scoring guidelines and notes contain examples of the most common approaches seen in student responses. These guidelines can be applied to alternate approaches to ensure that these alternate approaches are scored appropriately.

The function g has derivatives of all orders for all real numbers. The Maclaurin series for g is given by $g(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{2e^n + 3}$ on its interval of convergence.

Model Solution	Scoring	
(a) State the conditions necessary to use the integral test to determine convergence of the series $\sum_{n=0}^{\infty} \frac{1}{e^n}$. Use the integral test to show that $\sum_{n=0}^{\infty} \frac{1}{e^n}$ converges.		
e^{-x} is positive, decreasing, and continuous on the interval $[0, \infty)$.	Conditions	**1 point**
To use the integral test to show that $\sum_{n=0}^{\infty} \frac{1}{e^n}$ converges, show that $\int_0^{\infty} e^{-x}\,dx$ is finite (converges).	Improper integral	**1 point**
$\int_0^{\infty} e^{-x}\,dx = \lim_{b\to\infty} \int_0^{b} e^{-x}\,dx = \lim_{b\to\infty} \left(-e^{-x}\Big\vert_0^b\right) = \lim_{b\to\infty}\left(-e^{-b} + e^0\right) = 1$ Because the integral $\int_0^{\infty} e^{-x}\,dx$ converges, the series $\sum_{n=0}^{\infty} \frac{1}{e^n}$ converges.	Evaluation	**1 point**

Scoring notes:

- To earn the first point a response must list all three conditions: e^{-x} is positive, decreasing, and continuous.
- The second point is earned for correctly writing the improper integral or for presenting a correct limit equivalent to the improper integral (for example, $\lim\limits_{b\to\infty}\int_0^b e^{-x}\,dx$).
- To earn the third point a response must correctly use limit notation to evaluate the improper integral, find an evaluation of e^0 (or 1), and conclude that the integral converges or that the series converges.
- If an incorrect lower limit of 1 is used in the improper integral, then the second point is not earned. In this case, if the correct limit ($1/e$) is presented, then the response is eligible for the third point.
- If the response only relies on using a geometric series approach, then no points are earned [0-0-0].
- A response that presents an evaluation with ∞, such as $e^{-\infty} = 0$, does not earn the third point.

Total for part (a) 3 points

(b) Use the limit comparison test with the series $\sum_{n=0}^{\infty} \frac{1}{e^n}$ to show that the series $g(1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2e^n + 3}$ converges absolutely.

$$\lim_{n\to\infty} \frac{\frac{1}{e^n}}{\left\lvert \frac{(-1)^n}{2e^n + 3} \right\rvert} = \lim_{n\to\infty} \frac{2e^n + 3}{e^n} = 2$$ The limit exists and is positive. Therefore, because the series $\sum_{n=0}^{\infty} \frac{1}{e^n}$ converges, the series $\sum_{n=0}^{\infty} \left\lvert \frac{(-1)^n}{2e^n + 3} \right\rvert$ converges by the limit comparison test. Thus, the series $g(1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2e^n + 3}$ converges absolutely.	Sets up limit comparison	**1 point**
	Explanation	**1 point**

Scoring notes:

- The first point is earned for setting up the limit comparison, with or without absolute values. Limit notation is required to earn this point.
- The reciprocal of the given ratio is an acceptable alternative; the limit in this case is $1/2$.
- The second point cannot be earned without the use of absolute value symbols, which can occur explicitly or implicitly (e.g., a response might set up the limit comparison initially as $\lim_{n\to\infty} \frac{\frac{1}{e^n}}{\frac{1}{2e^n + 3}}$).
- Earning the second point requires correctly evaluating the limit and noting that the limit is a positive number. For example, $L = 2 > 0$ or $L = 1/2 > 0$. Therefore, comparing the limit L to 1 does not earn the explanation point.
- A response does not have to repeat that $\sum_{n=0}^{\infty} \frac{1}{e^n}$ converges.
- A response that draws a conclusion based only on the sequence (such as $\frac{1}{e^n}$) without referencing a series does not earn the second point.
- If the response does not explicitly use the limit comparison test, then no points are earned in this part.
- A response cannot earn the second point for just concluding that "the series" converges absolutely because there are multiple series in this part of the problem. The response must specify that the series $g(1)$ or $\sum_{n=0}^{\infty} \frac{(-1)^n}{2e^n + 3}$ converges absolutely.

Total for part (b) 2 points

(c) Determine the radius of convergence of the Maclaurin series for g.

$\left\lvert \dfrac{(-1)^{n+1} x^{n+1}}{2e^{n+1}+3} \cdot \dfrac{2e^{n}+3}{(-1)^{n} x^{n}} \right\rvert = \left\lvert \dfrac{\left(2e^{n}+3\right)x^{n+1}}{\left(2e^{n+1}+3\right)x^{n}} \right\rvert = \dfrac{2e^{n}+3}{2e^{n+1}+3}\lvert x\rvert$	Sets up ratio	**1 point**
$\lim\limits_{n\to\infty} \dfrac{2e^{n}+3}{2e^{n+1}+3}\lvert x\rvert = \dfrac{1}{e}\lvert x\rvert$	Computes limit of ratio	**1 point**
$\dfrac{1}{e}\lvert x\rvert < 1 \Rightarrow \lvert x\rvert < e$ The radius of convergence is $R = e$.	Answer	**1 point**

Scoring notes:

- The first point is earned for $\dfrac{(-1)^{n+1} x^{n+1}}{2e^{n+1}+3} \cdot \dfrac{2e^{n}+3}{(-1)^{n} x^{n}}$ or the equivalent. Once earned, this point cannot be lost.
- The second point cannot be earned without the first point.
- To be eligible for the third point the response must have found a limit for a presented ratio such that the limiting value of the coefficient on $\lvert x\rvert$ is finite and not 0. The third point is earned for setting up an inequality such that the limit is less than 1, solving for $\lvert x\rvert$, and interpreting the result to find the radius of convergence.
- The radius of convergence must be explicitly presented, for example, $R = e$. The third point cannot be earned by presenting an interval, for example $-e < x < e$, with no identification of the radius of convergence.

Total for part (c) 3 points

(d) The first two terms of the series $g(1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2e^n + 3}$ are used to approximate $g(1)$. Use the alternating series error bound to determine an upper bound on the error of the approximation.

The terms of the alternating series $g(1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2e^n + 3}$ decrease in magnitude to 0. The alternating series error bound for the error of the approximation is the absolute value of the third term of the series. Error $\le \left\lvert \frac{(-1)^2}{2e^2 + 3} \right\rvert = \frac{1}{2e^2 + 3}$	Answer	**1 point**

Scoring notes:

- A response of $\frac{1}{2e^2 + 3}$ earns this point.

Total for part (d) 1 point

Total for question 6 9 points