2022

AP® CollegeBoard

AP® Calculus BC

Sample Student Responses and Scoring Commentary

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Free-Response Question 2

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Part A (BC): Graphing calculator required

Question 2 — 9 points

General Scoring Notes
The model solution is presented using standard mathematical notation.

Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be correct to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.

A particle moving along a curve in the xy-plane is at position $(x(t), y(t))$ at time $t > 0$. The particle moves in such a way that $\frac{dx}{dt} = \sqrt{1 + t^2}$ and $\frac{dy}{dt} = \ln\left(2 + t^2\right)$. At time $t = 4$, the particle is at the point $(1, 5)$.

Model Solution	Scoring	
(a) Find the slope of the line tangent to the path of the particle at time $t = 4$.		
$\left.\frac{dy}{dx}\right\|_{t=4} = \frac{y'(4)}{x'(4)} = \frac{\ln 18}{\sqrt{17}} = 0.701018$ The slope of the line tangent to the path of the particle at time $t = 4$ is 0.701.	Answer	**1 point**

Scoring notes:

- To earn the point, the setup used to perform the calculation must be evident in the response. The following examples earn the point: $\frac{y'(4)}{x'(4)} = 0.701$, $\frac{\ln\left(2 + 4^2\right)}{\sqrt{1 + 4^2}}$, or $\frac{\ln 18}{\sqrt{17}}$.
- Note: A response with an incorrect equation of the form "function = constant", such as $\frac{y'(t)}{x'(t)} = \frac{\ln(18)}{\sqrt{17}}$, will not earn the point. However, such a response will be eligible for any points for similar errors in subsequent parts.

Total for part (a) **1 point**

(b) Find the speed of the particle at time $t = 4,$ and find the acceleration vector of the particle at time $t = 4.$

$\sqrt{(x'(4))^2 + (y'(4))^2} = \sqrt{17 + (\ln 18)^2} = 5.035300$ The speed of the particle at time $t = 4$ is 5.035.	Speed	**1 point**
$a(4) = \langle x''(4), y''(4) \rangle = \left\langle \frac{4}{\sqrt{17}}, \frac{4}{9} \right\rangle = \langle 0.970143, 0.444444 \rangle$ The acceleration vector of the particle at time $t = 4$ is $\langle 0.970, 0.444 \rangle.$	First component of acceleration	**1 point**
	Second component of acceleration	**1 point**

Scoring notes:

- To earn any of these points, the setup used to perform the calculation must be evident in the response. For example, $\sqrt{(x'(4))^2 + (y'(4))^2} = 5.035$ or $\sqrt{17 + (\ln 18)^2}$ earns the first point, and $\langle x''(4), y''(4) \rangle = \left\langle \frac{4}{\sqrt{17}}, \frac{4}{9} \right\rangle$ earns both the second and third points.
- The second and third points can be earned independently.
- If the acceleration vector is not presented as an ordered pair, the x- and y-components must be labeled.
- If the components of the acceleration vector are reversed, the response does not earn either of the last 2 points.
- A response which correctly calculates expressions for both $x''(t) = \frac{t}{\sqrt{1 + t^2}}$ and $y''(t) = \frac{2t}{2 + t^2},$ but which fails to evaluate both of these expressions at $t = 4,$ earns only 1 of the last 2 points.
- An unsupported acceleration vector earns only 1 of the last 2 points.

Total for part (b) 3 points

(c) Find the y-coordinate of the particle's position at time $t = 6$.

$y(6) = y(4) + \int_4^6 \ln\left(2 + t^2\right) dt$	Integrand	**1 point**
	Uses $y(4)$	**1 point**
$= 5 + 6.570517 = 11.570517$ The y-coordinate of the particle's position at time $t = 6$ is 11.571 (or 11.570).	Answer	**1 point**

Scoring notes:

- For the first point, an integrand of $\ln\left(2 + t^2\right)$ can appear in either an indefinite integral or an incorrect definite integral.
- A definite integral with incorrect limits is not eligible for the answer point.
- Similarly, an indefinite integral is not eligible for the answer point.
- For the second point, the value for $y(4)$ must be added to a definite integral.
- A response that reports the correct x-coordinate of the particle's position at time $t = 6$ as $x(6) = x(4) + \int_4^6 \sqrt{1 + t^2}\, dt = 11.200$ (or 11.201) instead of the y-coordinate, earns 2 out of the 3 points.
- A response that earns the first point but not the second can earn the third point with an answer of 6.571 (or 6.570).
- If the differential is missing:
  - $y(6) = \int_4^6 \ln\left(2 + t^2\right)$ earns the first point and is eligible for the third.
  - $y(6) = \int_4^6 \ln\left(2 + t^2\right) + y(4)$ does not earn the first point but is eligible for the second and third points in the presence of the correct answer.
  - $y(6) = y(4) + \int_4^6 \ln\left(2 + t^2\right)$ earns the first two points and is eligible for the third.

Total for part (c) 3 points

(d) Find the total distance the particle travels along the curve from time $t = 4$ to time $t = 6$.

$\int_4^6 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$	Integrand	**1 point**
$= 12.136228$ The total distance the particle travels along the curve from time $t = 4$ to time $t = 6$ is 12.136.	Answer	**1 point**

Scoring notes:

- The first point is earned for presenting the correct integrand in a definite integral.
- To earn the second point, a response must have earned the first point and must present the value 12.136.
- An unsupported answer of 12.136 does not earn either point.

Total for part (d) 2 points

Total for question 2 9 points

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Answer QUESTION 2 parts (a) and (b) on this page.

Response for question 2(a)

$$\frac{dx}{dt} = \sqrt{1+t^2}$$

$$\frac{dy}{dt} = \ln(2+t^2)$$

$$\frac{dy}{dx} = \frac{\ln(2+t^2)}{\sqrt{1+t^2}}$$

slope of line tangent to particle's path

↓

↳ plug in 4 for t in calc: 0.7016

Response for question 2(b)

$$\text{speed} = \sqrt{(x'(t))^2 + (y'(t))^2}$$

$$\hookrightarrow \sqrt{\left(\sqrt{1+t^2}\right)^2 + \left(\ln(2+t^2)\right)^2}$$

speed of the particle

↓

↳ plug in 4 for t on calculator: 5.0353

Acceleration vector

$$\left\langle \frac{d}{dt}\left(\sqrt{1+t^2}\right), \frac{d}{dt}\left(\ln(2+t^2)\right) \right\rangle \rightarrow \text{plug in } 4 \rightarrow \langle 0.9701, 0.4444 \rangle$$

Use a pencil or a pen with black or dark blue ink. Do NOT write your name. Do NOT write outside the box.

Answer QUESTION 2 parts (c) and (d) on this page.

Response for question 2(c)

at $t=4$, particle is at $(1,5)$

$$5 + \int_4^6 \ln(2+t^2)\,dt$$

$\uparrow$

$\frac{dy}{dt}$

at time $t=6$, the y-coordinate of the particle's position is

$\boxed{11.5705}$

Response for question 2(d)

total distance = magnitude of speed

$$\int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\,dt$$

$$\int_4^6 \sqrt{\left(\sqrt{1+t^2}\right)^2 + (\ln(2+t^2))^2}\,dt$$

The particle travels

↳ in calculator → 12.1362 from

$t=4$ to $t=6$

2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2

Answer QUESTION 2 parts (a) and (b) on this page.

Response for question 2(a)

$$\frac{\frac{dy}{dt}}{\frac{dx}{dt}} \qquad \frac{\ln(2+t^2)}{\sqrt{1+t^2}} \text{ at } t=4$$

$$\text{slope} = \boxed{0.701}$$

Response for question 2(b)

$$\sqrt{\left(\sqrt{1+t^2}\right)^2 + \left(\ln(2+t^2)\right)^2} = \text{ at } t=4$$

$$\text{speed} = \boxed{5.0353}$$

$$\sqrt{\left(\sqrt{1+t^2}\right)'^{\,2} + \left(\ln(2+t^2)\right)'^{\,2}}$$

$$(0.970142)^2 + (0.444)^2 \text{ at } t=4$$

$$= \boxed{1.0671} \text{ acceleration}$$

Use a pencil or a pen with black or dark blue ink. Do NOT write your name. Do NOT write outside the box.

Answer QUESTION 2 parts (c) and (d) on this page.

Response for question 2(c)

~~$\int \sqrt{1+t^2}$~~

$\int_4^6 \ln(2+t^2)\,dt$

at $t=4$

$(1,5)$

$= 6.5705 + 5$

$\boxed{y\text{-coord} = 11.5705}$

Response for question 2(d)

$\int_4^6 \sqrt{1+t^2}\,dt$

$= 10.2006$

from $t=4$ to $t=6$

$(1,5) \rightarrow (11.2006, 11.5705)$

$5.099 \rightarrow 16.1037$

vector difference

$\boxed{11.0047}$

Answer QUESTION 2 parts (a) and (b) on this page.

Response for question 2(a)

$$\frac{dr}{dt} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\ln(2+t^2)}{\sqrt{1+t^2}} \quad \text{at } t=4$$

$$\frac{\ln(2+16)}{\sqrt{1+16}} = \frac{\ln 18}{\sqrt{17}} = 0.701$$

Response for question 2(b)

$$\frac{dr}{dt} = \frac{dy/dt}{dx/dt} = \frac{\ln(2+t^2)}{\sqrt{1+t^2}} \quad \text{at } t=4 \quad = \underset{\substack{\wedge \\ \text{speed}}}{0.701}$$

$$\frac{d^2r}{dt^2} = \frac{\frac{1}{2+t^2}\cdot 2t\cdot\sqrt{1+t^2} - \left(-\frac{1}{\cancel{2}}\cdot\frac{1}{\sqrt{1+t^2}}\cdot\cancel{2}t\right)\left(\ln(2+t^2)\right)}{1+t^2}$$

$$= \frac{\frac{2t\sqrt{1+t^2}}{2+t^2} + \frac{t\ln(2+t^2)}{\sqrt{1+t^2}}}{t^2} = \frac{\frac{8\sqrt{17}}{18} + \frac{4\ln 8}{\sqrt{17}}}{16} = \underset{\substack{\uparrow \\ \text{acceleration}}}{0.241}$$

Use a pencil or a pen with black or dark blue ink. Do NOT write your name. Do NOT write outside the box.

Answer QUESTION 2 parts (c) and (d) on this page.

Response for question 2(c)

$$\frac{dy}{dt} = \frac{\ln(2+t^2)}{1}$$

$$\int 1 \, dy = \int \ln(2+t^2)\, dt$$

$$y = \int \ln(2+t^2)\, dt$$

$$\int_4^6 \ln(2+t^2)\, dt$$

$$=$$

Response for question 2(d)

$$\int_4^6 \sqrt{1+\left(\frac{\ln(2+t^2)}{\sqrt{1+t^2}}\right)^2}\, dt = 2.383$$

Question 2

Note: Student samples are quoted verbatim and may contain spelling and grammatical errors.

Overview

In this problem a particle is moving along a curve in the xy-plane with position $(x(t), y(t))$. At time $t = 4,$ the particle is at the point $(1, 5)$. The particle moves so that $\frac{dx}{dt} = \sqrt{1 + t^2}$ and $\frac{dy}{dt} = \ln\left(2 + t^2\right)$.

In part (a) students were asked to find the slope of the line tangent to the path of the particle at time $t = 4$. A correct response would provide the setup $\left.\frac{dy}{dx}\right|_{t=4} = \frac{y'(4)}{x'(4)}$ and evaluate to find the slope is $\frac{\ln 18}{\sqrt{17}}$ or, using a calculator, 0.701.

In part (b) students were asked to find the speed of the particle and the acceleration vector of the particle, both at time $t = 4$. A correct response would indicate that the speed of the particle at this time is $\sqrt{(x'(4))^2 + (y'(4))^2} = 5.0353$ and the acceleration is $a(4) = \langle x''(4), y''(4) \rangle = \langle 0.970, 0.444 \rangle$. (Either or both of these answers could be provided without use of the calculator as $\sqrt{17 + (\ln 18)^2}$ or $\left\langle \frac{4}{\sqrt{17}}, \frac{4}{9} \right\rangle$, respectively.)

In part (c) students were asked to find the y-coordinate of the particle's position at time $t = 6$. A correct response would integrate the rate of change of the particle's y-position, $\frac{dy}{dt} = \ln\left(2 + t^2\right)$, from time $t = 4$ to time $t = 6$, then add the initial condition $y(4) = 5$ to find a y-coordinate of the particle's position of 11.571.

And in part (d) students were asked to find the total distance the particle travels along the curve from time $t = 4$ to time $t = 6$. A correct response would provide the calculator setup of the integral of the particle's speed over this time interval, then evaluate to find a total distance of 12.136.

Sample: 2A
Score: 9

The response earned 9 points: 1 point in part (a), 3 points in part (b), 3 points in part (c), and 2 points in part (d).

In part (a) the response earned the point with the correct presentation of the function $\frac{dy}{dx}$ in the third line and the correct evaluation of this function at $t = 4$ to produce the answer of 0.7010.

In part (b) the response earned the first point with the correct expression for the speed $\sqrt{(x'(t))^2 + (y'(t))^2}$ and a correct evaluation of this expression at $t = 4$ to produce the answer 5.0353. The second point is earned for the x-component in the last line with the expression $\frac{d}{dt}\left(\sqrt{1 + t^2}\right)$, the statement "plug in 4," and the correct answer of 0.9701. The third point is earned for the y-component in the last line with expression $\frac{d}{dt}\left(\ln\left(2 + t^2\right)\right)$, the statement "plug in 4," and the correct answer of 0.4444.

Question 2 (continued)

In part (c) the response earned the first point by using the expression $\ln\left(2 + t^2\right)$ as an integrand. Adding 5 to the associated definite integral earned the second point, and the circled answer of 11.5705 earned the third point.

In part (d) the response earned the first point with the definite integral in the third line and earned the second point with the correct answer of 12.1362.

Sample: 2B
Score: 5

The response earned 5 points: 1 point in part (a), 1 point in part (b), 3 points in part (c), and no points in part (d).

In part (a) the response earned the point by evaluating the function $\frac{\ln\left(2 + t^2\right)}{\sqrt{1 + t^2}}$ at $t = 4$ to produce the correct answer.

In part (b) the first point was earned with the correct equation for speed and the correct evaluation of this equation at $t = 4$ to produce the boxed answer in the second line. The second and third points were not earned because neither the x- nor the y-component of the acceleration vector is provided.

In part (c) the use of $\ln\left(2 + t^2\right)$ as an integrand earned the first point. The phrase "at $t = 4 \rightarrow (1, 5)$" pointing to the $+5$ in the second line earned the second point, and the correct value for the y-coordinate earned the third point.

In part (d) the incorrect integrand did not earn the first point, and the incorrect boxed answer did not earn the second point.

Sample: 2C
Score: 2

The response earned 2 points: 1 point in part (a), no points in part (b), 1 point in part (c), and no points in part (d).

In part (a) the response would have earned the first point with the expression $\frac{\ln(2 + 16)}{\sqrt{1 + 16}}$ in the second line. The response simplified the expression correctly and earned the first point.

In part (b) the first point was not earned because the response computes the slope of the tangent line instead of the speed at $t = 4.$ The second and third points were not earned because the response provides neither the x- nor y-component of the acceleration vector.

In part (c) the response earned the first point with the indefinite integral $\int \ln\left(2+t^2\right) dt.$ The second point was not earned because $y(4) = 5$ is not used, and the third point was not earned because the correct answer is not given.

In part (d) the response did not earn the first point because the integrand of the definite integral is incorrect, and the second point was not earned because the answer is incorrect.