2022

AP® | CollegeBoard

AP® Calculus BC

Sample Student Responses and Scoring Commentary

Inside:

Free-Response Question 3

☑ Scoring Guidelines

☑ Student Samples

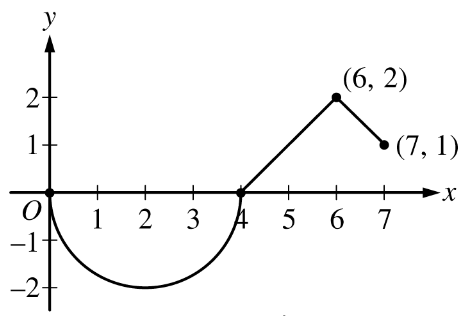
☑ Scoring Commentary

Part B (AB or BC): Graphing calculator not allowed
Question 3 — 9 points

General Scoring Notes

The model solution is presented using standard mathematical notation.

Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be correct to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.



Graph of f'

Let f be a differentiable function with $f(4) = 3$. On the interval $0 \le x \le 7$, the graph of f', the derivative of f, consists of a semicircle and two line segments, as shown in the figure above.

	Model Solution	Scoring	
(a)	Find $f(0)$ and $f(5)$.		
	$f(0) = f(4) + \int_4^0 f'(x)\,dx = 3 - \int_0^4 f'(x)\,dx = 3 + 2\pi$ $f(5) = f(4) + \int_4^5 f'(x)\,dx = 3 + \frac{1}{2} = \frac{7}{2}$	Area of either region – OR – $\int_0^4 f'(x)\,dx$ – OR – $\int_4^5 f'(x)\,dx$	**1 point**
		$f(0)$	**1 point**
		$f(5)$	**1 point**

Scoring notes:

- A response with answers of only $f(0) = \pm 2\pi$, or only $f(5) = \frac{1}{2}$, or both earns 1 of the 3 points.
- A response displaying $f(5) = \frac{7}{2}$ and a missing or incorrect value for $f(0)$ earns 2 of the 3 points.
- The second and third points can be earned in either order.
- Read unlabeled values from left to right and from top to bottom as $f(0)$ and $f(5)$. A single value must be labeled as either $f(0)$ or $f(5)$ in order to earn any points.

Total for part (a) — 3 points

(b) Find the x-coordinates of all points of inflection of the graph of f for $0 < x < 7$. Justify your answer.

The graph of f has a point of inflection at each of $x = 2$ and $x = 6$, because $f'(x)$ changes from decreasing to increasing at $x = 2$ and from increasing to decreasing at $x = 6$.	Answer	**1 point**
	Justification	**1 point**

Scoring notes:

- A response that gives only one of $x = 2$ or $x = 6$, along with a correct justification, earns 1 of the 2 points.
- A response that claims that there is a point of inflection at any value other than $x = 2$ or $x = 6$ earns neither point.
- To earn the second point a response must use correct reasoning based on the graph of f'. Examples of correct reasoning include:
  - Correctly discussing the signs of the slopes of the graph of f'
  - Citing $x = 2$ and $x = 6$ as the locations of local extrema on the graph of f'
- Examples of reasoning not (sufficiently) connected to the graph of f' include:
  - Reasoning based on sign changes in f'' unless the connection is made between the sign of f'' and the slopes of the graph of f'
  - Reasoning based only on the concavity of the graph of f
- The second point cannot be earned by use of vague or undefined terms such as "it" or "the function" or "the derivative."
- Responses that report inflection points as ordered pairs must report the points $(2, 3 + \pi)$ and $(6, 5)$ in order to earn the first point. If the y-coordinates are reported incorrectly, the response remains eligible for the second point.

Total for part (b) 2 points

(c) Let g be the function defined by $g(x) = f(x) - x$. On what intervals, if any, is g decreasing for $0 \le x \le 7$? Show the analysis that leads to your answer.

$g'(x) = f'(x) - 1$	$g'(x) = f'(x) - 1$	**1 point**
$f'(x) - 1 \le 0 \Rightarrow f'(x) \le 1$ The graph of g is decreasing on the interval $0 \le x \le 5$ because $g'(x) \le 0$ on this interval.	Interval with reason	**1 point**

Scoring notes:

- The first point can be earned for $f'(x) \le 1$ or the equivalent, in words or symbols.
- Endpoints do not need to be included in the interval to be eligible for the second point.

Total for part (c) 2 points

(d) For the function g defined in part (c), find the absolute minimum value on the interval $0 \le x \le 7$. Justify your answer.

g is continuous, $g'(x) < 0$ for $0 < x < 5$, and $g'(x) > 0$ for $5 < x < 7$. Therefore, the absolute minimum occurs at $x = 5$, and $g(5) = f(5) - 5 = \frac{7}{2} - 5 = -\frac{3}{2}$ is the absolute minimum value of g.	Considers $g'(x) = 0$	**1 point**
	Answer with justification	**1 point**

Scoring notes:

- A justification that uses a local argument, such as "g' changes from negative to positive (or g changes from decreasing to increasing) at $x = 5$" must also state that $x = 5$ is the only critical point.
- If $g'(x) = 0$ (or equivalent) is not declared explicitly, a response that isolates $x = 5$ as the only critical number belonging to $(0, 7)$ earns the first point.
- A response that imports $g'(x) = f'(x)$ from part (c) is eligible for the first point but not the second.
  - In this case, consideration of $x = 4$ as the only critical number on $(0, 7)$ earns the first point.
- Solution using Candidates Test:

  $g'(x) = f'(x) - 1 = 0 \Rightarrow x = 5,\ x = 7$

x	$g(x)$
0	$3 + 2\pi$
5	$-\frac{3}{2}$
7	$-\frac{1}{2}$

  The absolute minimum value of g on the interval $0 \le x \le 7$ is $-\frac{3}{2}$.

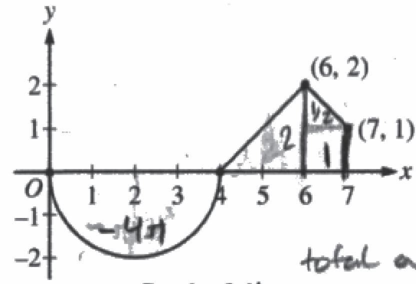
- When using a Candidates Test, a response may import an incorrect value of $f(0) = g(0) > -\frac{3}{2}$ from part (a). The second point can only be earned for an answer of $-\frac{3}{2}$.

Total for part (d) 2 points

Total for question 3 9 points

3 3 3 3 3 NO CALCULATOR ALLOWED 3 3 3 3 3

Answer QUESTION 3 parts (a) and (b) on this page.



Graph of f'

$f(4)=3$

total area: $3.5-4\pi$

Response for question 3(a)

find $f(0)$ & $f(5)$

$\int_4^0 f'(x)\,dx = f(0)-f(4)$

$-\int_0^4 f'(x)\,dx = f(0)-f(4)$

$4\pi = f(0)-3$

$f(0)=4\pi+3$

$\int_4^5 f'(x)\,dx = f(5)-f(4)$

$\frac{1}{2} = f(5)-3$

$f(5)=3.5$

Response for question 3(b)

POI: $x=2$, 6
$x=6$

a point of inflection occurs when the second derivative changes sign. The graph is a graph of f', ~~and f'' is the slope~~ of f', so wherever the graph changes from increasing to decreasing, or the vice versa is a point of inflection

3 3 3 3 3 **NO CALCULATOR ALLOWED** 3 3 3 3 3

Answer QUESTION 3 parts (c) and (d) on this page.

Response for question 3(c)

$g'(x) = f'(x) - 1$

$f'(x) - 1 = 0$

$f'(x) = 1$

$x = 5 \quad x = 7$

decreasing in interval (0, 5), because $g'(x) = f'(x) - 1$ is negative in this interval, meaning g is decreasing.

Response for question 3(d)

crit points: $x = 5 \quad x = 7$

endpoints: $x = 0 \quad x = 7$

$g(0) = f(0) - 0 = 4\pi + 3$

$g(5) = f(5) - 5 = 3.5 - 5 = -1.5$

$g(7) = f(7) - 7 = 6.5 - 7 = -0.5$

absolute minimum: $(5, -1.5)$

$\int_4^7 f'(x)\,dx = f(7) - f(4)$

$3.5 = f(7) - f(4)$

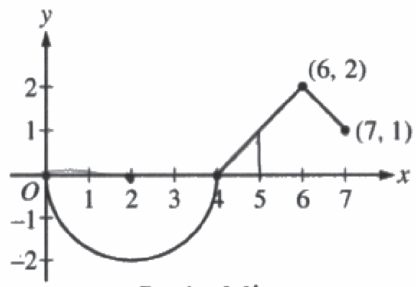
$3.5 = f(7) - 3$

$f(7) = 6.5$

The abs min of $g(x)$ is -1.5 based on the values found through the critical points of the function

3 3 3 3 3 NO CALCULATOR ALLOWED 3 3 3 3 3

Answer QUESTION 3 parts (a) and (b) on this page.



Graph of f'

Response for question 3(a)

$f(4)=3$

using FTC:

$\int_0^4 f'(x) = f(4) - f(0)$

$\int_0^4 f'(x) = \frac{1}{2}\pi(2)^2 = -2\pi$

$-2\pi = 3 - f(0)$

$f(0) = 3 + 2\pi$

$\int_4^5 f'(x) = f(5) - f(4)$

$\int_4^5 f'(x) = \frac{1}{2}(1\times1) = \frac{1}{2}$

$\frac{1}{2} = f(5) - 3$

$f(5) = \frac{7}{2}$

Response for question 3(b)

point of inflection is where $f''(x) = 0$ (horizontal tangent)

There is a horizontal tangent at the bottom of the semicircle, at $x = 2$.

Use a pencil or a pen with black or dark blue ink. Do NOT write your name. Do NOT write outside the box.

Q5106/8

Answer QUESTION 3 parts (c) and (d) on this page.

Response for question 3(c)

$g(x) = f(x) - x$

$g'(x) = f'(x) - 1$

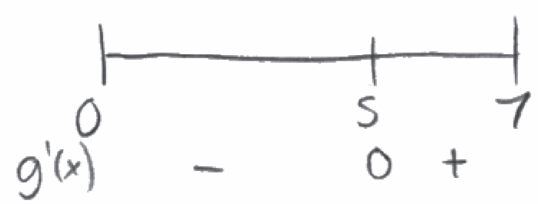
g is decreasing when $g'(x) < 0$

$g'(x) < 0$ when $f'(x) < 1$

$f'(x) < 1$ on the interval $0 < x < 5$

Response for question 3(d)

minimum is where the first derivative goes from negative to positive.

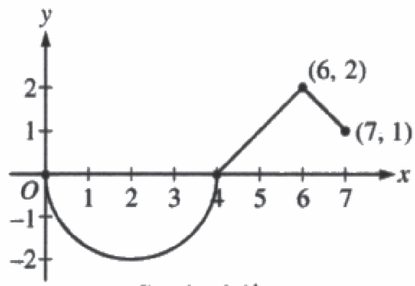


$g'(x)$ goes from negative to positive at $x = 5$

$g(5) = f(5) - (5)$ $g(5) = 7/2 - 5 = -3/2$ units

3 3 3 3 3 NO CALCULATOR ALLOWED 3 3 3 3 3

Answer QUESTION 3 parts (a) and (b) on this page.



Graph of f'

Response for question 3(a)

$f(0) = \int_0^0 f'(x) = 0$ $f(0) = 0$

$f(5) = \int_0^5 f'(x) = \frac{1}{2} - 2\pi$ $f(5) = \frac{1}{2} - 2\pi$

$-\frac{1}{2}\pi r^2$

$-\frac{1}{2}4\pi$ $-2\pi + \frac{1}{2}$

Response for question 3(b)

$x=0$ At $x=0$ and $x=4$, the first derivative
$x=4$ switches signs, meaning there is a zero, or a point of inflection for the second derivative.

3 3 3 3 3 NO CALCULATOR ALLOWED 3 3 3 3 3

Answer QUESTION 3 parts (c) and (d) on this page.

Response for question 3(c)

g is dec on the intervals (0,7) this is because g is negative along the whole graph, making it decreasing on the whole graph.

Response for question 3(d)

$g'(x) = f'(x) - 1$

$f'(x) - 1 = 0$

$x = 5$

abs min at $x = 5$

Use a pencil or a pen with black or dark blue ink. Do NOT write your name. Do NOT write outside the box.

Q5113/9

Question 3

Note: Student samples are quoted verbatim and may contain spelling and grammatical errors.

Overview

In this problem the graph of a function f', which consists of a semicircle and two line segments on the interval $0 \le x \le 7$, is provided. It is also given that this is the graph of the derivative of a differentiable function f with $f(4) = 3$.

In part (a) students were asked to find $f(0)$ and $f(5)$. To find $f(0)$ a correct response uses geometry and the Fundamental Theorem of Calculus to calculate the signed area of the semicircle, $\int_0^4 f'(x)\,dx = -2\pi$, and subtracts this value from the initial condition, $f(4) = 3$, to obtain a value of $3 + 2\pi$. To find $f(5)$ a correct response would add the initial condition to the signed area $\int_4^5 f'(x)\,dx = \frac{1}{2}$, found using geometry, to obtain a value of $\frac{7}{2}$.

In part (b) students were asked to find the x-coordinates of all points of inflection on the graph of f for $0 < x < 7$ and to justify their answers. A correct response would use the given graph to determine that the graph of $f'(x)$ changes from decreasing to increasing, or vice versa, at the points $x = 2$ and $x = 6$. Therefore, these are the inflection points of the graph of f.

In part (c) students were told that $g(x) = f(x) - x$ and are asked to determine on which intervals, if any, the function g is decreasing. A correct response would find that $g'(x) = f'(x) - 1$ and then use the given graph of f' to determine that when $0 \le x \le 5$, $f'(x) \le 1 \Rightarrow g'(x) \le 0$. Therefore, g is decreasing on the interval $0 \le x \le 5$.

In part (d) students were asked to find the absolute minimum value of $g(x) = f(x) - x$ on the interval $0 \le x \le 7$. A correct response would use the work from part (c) to conclude $g'(x) < 0$ for $0 < x < 5$ and $g'(x) > 0$ for $5 < x < 7$. Thus the absolute minimum of g occurs at $x = 5$. Using the work from part (a), which found the value of $f(5)$, the absolute minimum value of g is $g(5) = f(5) - 5 = \frac{7}{2} - 5 = -\frac{3}{2}$.

Sample: 3A
Score: 8

The response earned 8 points: 2 points in part (a), 2 points in part (b), 2 points in part (c), and 2 points in part (d).

In part (a) the first point was earned for the integral expression to the left of the first equal sign. The second point was not earned because the value of $f(0)$ is reported incorrectly. The third point was earned because the stated value of $f(5)$ is correct.

In part (b) the first point was earned for the boxed answer on lines 1 and 2, which declares the "POI" as $x = 2$ and $x = 6$. The second point was earned for the explanation given in the paragraph below the boxed answer. The response appeals to the change in sign of f'', which alone would not be sufficient but goes on to declare that "f'' is the slope of f'" and indicates correctly that where the graph (of f') "changes from increasing to decreasing, or vice versa is a point of inflection."

Question 3 (continued)

In part (c) the first point was earned on line 1 for $g'(x) = f'(x) - 1.$ The second point was earned on lines 1, 2, and 3 on the right for giving the correct interval $(0, 5)$ with the reason that "$g'(x) = f'(x) - 1$ is negative in this interval."

In part (d) the first point was earned on line 1 for consideration of only $x = 5$ (and the endpoints) as possible locations of the absolute minimum. The second point was earned on line 8 for declaring -1.5 as the absolute minimum. A Candidates Test is carried out correctly, with an incorrect value of $f(0)$ that is greater than $-\frac{3}{2}$ imported from part (a). Such responses are still eligible to earn the second point as long as $g(0)$ is declared to be that same imported value, and there are no mistakes in reported values of $g(5)$ and $g(7)$.

Sample: 3B
Score: 6

The response earned 6 points: 3 points in part (a), no points in part (b), 2 points in part (c), and 1 point in part (d).

In part (a) the response earned the first point for $\int_0^4 f'(x)$ on line 3 on the left. Note that the integrals are missing the differential dx; this oversight is not penalized. The second and third points were earned for the boxed, correct values of $f(0)$ and $f(5)$.

In part (b) the response did not earn the first point, because $x = 6$ is not given among the answers. The second point was not earned because the reasoning that "there is a horizontal tangent at the bottom of the semicircle" is not sufficient. A response must make a specific appeal to f' in order for the second point to be earned.

In part (c) the response earned the first point on line 2 for the correct derivative of $g'(x).$ The response earned the second point for the correct boxed interval on line 5, with correct reasoning on lines 3 and 4.

In part (d) the response earned the first point for consideration of a sign change in $g'(x)$ below the sign chart. Although the response does have the correct answer of $-\frac{3}{2},$ it uses the local argument that "$g'(x)$ goes from negative to positive at $x = 5$" without an appeal to the whole interval $(0, 7).$ Therefore, the second point was not earned. The response appears, perhaps, to address the interval with a sign chart, but the response must explain the conclusions gathered from the chart in order for the point to be earned.

Sample: 3C
Score: 2

The response earned 2 points: 1 point in part (a), no points in part (b), no points in part (c), and 1 point in part (d).

In part (a) the response earned the first point on line 2 for consideration of $\frac{1}{2}$ (the area of the necessary triangular region) to the right of the second equal sign. The response did not earn the second and third points because the answers for $f(0)$ and $f(5)$ are both incorrect.

In part (b) the response did not earn either point because answers other than $x = 2$ or $x = 6$ are given (in this case, $x = 0$ and $x = 4$), which renders the response ineligible for either point.

Question 3 (continued)

In part (c) the response did not earn the first point because $g'(x)$ (or equivalent) is not considered. It is not possible to earn the second point without having earned the first point in part (c).

In part (d) the response earned the first point on line 2 for "$f'(x) - 1 = 0.$" No function value at $x = 5$ is reported; thus, the second point was not earned.