2022

AP CollegeBoard

AP® Calculus BC

Sample Student Responses and Scoring Commentary

Inside:

Free-Response Question 5

- ☑ **Scoring Guidelines**
- ☑ **Student Samples**
- ☑ **Scoring Commentary**

Part B (BC): Graphing calculator not allowed
Question 5 — 9 points

General Scoring Notes
The model solution is presented using standard mathematical notation.

Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be correct to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.

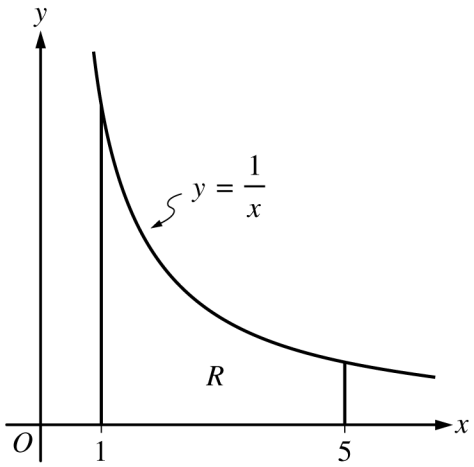

Figure 1

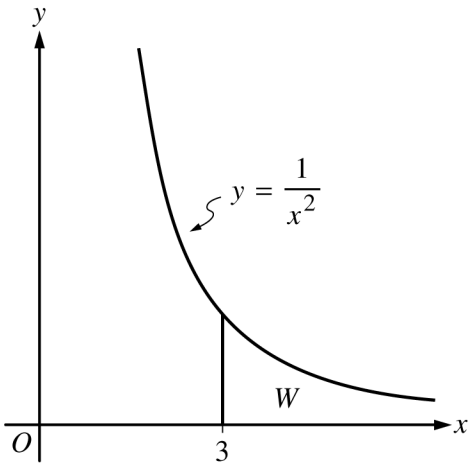

Figure 2

Figures 1 and 2, shown above, illustrate regions in the first quadrant associated with the graphs of $y = \frac{1}{x}$ and $y = \frac{1}{x^2}$, respectively. In Figure 1, let R be the region bounded by the graph of $y = \frac{1}{x}$, the x-axis, and the vertical lines $x = 1$ and $x = 5$. In Figure 2, let W be the unbounded region between the graph of $y = \frac{1}{x^2}$ and the x-axis that lies to the right of the vertical line $x = 3$.

	Model Solution	Scoring	
(a)	Find the area of region R.		
	$\text{Area} = \int_1^5 \frac{1}{x}\,dx$	Integral	**1 point**
	$= \ln x\big\|_1^5$ $= \ln 5 - \ln 1 = \ln 5$	Answer	**1 point**

Scoring notes:

- A definite integral with incorrect bounds does not earn either point.
- An unevaluated indefinite integral does not earn either point.
- An indefinite integral that is evaluated in a later step may earn one or both points. For example, $\int \frac{1}{x}\,dx = \ln 5 - \ln 1$ (or $\ln 5$) does not earn the first point but does earn the second. However, $\int \frac{1}{x}\,dx = \ln x + C \Rightarrow \text{Area} = \ln 5 - \ln 1$ earns both points.

Total for part (a) 2 points

(b) Region R is the base of a solid. For the solid, at each x the cross section perpendicular to the x-axis is a rectangle with area given by $xe^{x/5}$. Find the volume of the solid.

$\text{Volume} = \int_1^5 xe^{x/5}\,dx$	Definite integral	**1 point**
Using integration by parts, $u = x \quad dv = e^{x/5}\,dx$ $du = dx \quad v = 5e^{x/5}$	u and dv	**1 point**
$\int xe^{x/5}\,dx = 5xe^{x/5} - \int 5e^{x/5}\,dx$ $= 5xe^{x/5} - 25e^{x/5} + C$ $= 5e^{x/5}(x-5) + C$	$\int xe^{x/5}\,dx$ $= 5xe^{x/5} - \int 5e^{x/5}\,dx$	**1 point**
$\text{Volume} = 5e^{x/5}(x-5)\Big\|_1^5$ $= 5e(0) - 5e^{1/5}(-4) = 20e^{1/5}$	Answer	**1 point**

Scoring notes:

- The first point is earned for $c\int_1^5 xe^{x/5}\,dx$, where $c \neq 0$. Errors of $c \neq 1$, for example $c = \pi$, will not earn the fourth point.
- Incorrect integrals that require integration by parts are still eligible for the second and third points. Both of these points will be earned with at least one correct application of integration by parts.
- The second point will be earned with an implied u and dv in the presence of $5xe^{x/5} - \int 5e^{x/5}\,dx$.
- The tabular method may be used to show integration by parts. In this case, the second point is earned by having columns (labeled or unlabeled) that begin with x and $e^{x/5}$. The third point is earned for either $5xe^{x/5} - \int 5e^{x/5}\,dx$ or $5xe^{x/5} - 25e^{x/5}$.
- Limits of integration may be present, omitted, or partially present in the work for the second and third points.
- The fourth point is earned only for the correct answer.

Total for part (b) 4 points

(c) Find the volume of the solid generated when the unbounded region W is revolved about the x-axis.

$\text{Volume} = \pi\displaystyle\int_3^\infty \left(\frac{1}{x^2}\right)^2 dx = \pi \lim_{b\to\infty}\int_3^b \frac{1}{x^4}\,dx$	Improper integral	**1 point**
$= \pi \lim_{b\to\infty}\left(\left.\frac{1}{-3x^3}\right\|_3^b\right)$	Antiderivative	**1 point**
$= \pi \lim_{b\to\infty}\left(\frac{1}{-3}\right)\left[\frac{1}{b^3} - \frac{1}{3^3}\right]$ $= \pi\left(\frac{1}{-3}\right)\left(0 - \frac{1}{3^3}\right) = \frac{\pi}{81}$	Answer	**1 point**

Scoring notes:

- The first point is earned for either $c\displaystyle\int_3^\infty \left(\frac{1}{x^2}\right)^2 dx$ or $\lim_{b\to\infty} c\int_3^b \frac{1}{x^4}\,dx$, where $c \neq 0$. Errors of $c \neq \pi$ will not earn the third point.
- The second point is earned for a correct antiderivative of any integrand of the form $\frac{1}{x^n}$, for any integer $n \geq 2$.
- To earn the answer point, a response must use correct limit notation and cannot include arithmetic with infinity, such as $\frac{1}{\infty^3}$.

Total for part (c) 3 points

Total for question 5 9 points

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Answer QUESTION 5 part (a) on this page.

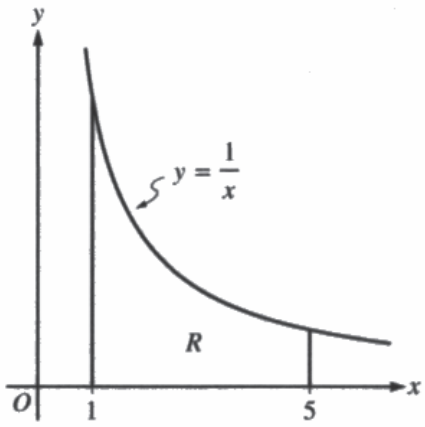


Figure 1

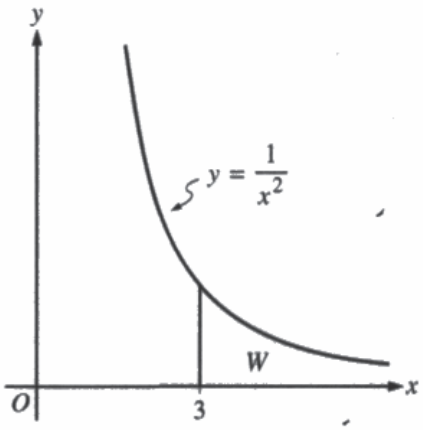


Figure 2

Response for question 5(a)

(a) area of $R = \int_1^5 \frac{1}{x}\,dx = [\ln|x|]_1^5 = \ln 5 - \ln 1 = \boxed{\ln 5}$

The area of region R is $\ln 5$.

5 5 5 5 5 NO CALCULATOR ALLOWED 5 5 5 5 5

Answer QUESTION 5 parts (b) and (c) on this page.

Response for question 5(b)

$dV = xe^{x/5}\,dx$

$V = \int_1^5 xe^{x/5}\,dx$

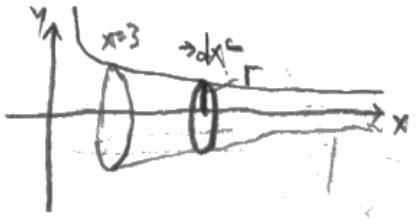
$u = x \quad dv = e^{x/5}\,dx$

$du = dx \quad v = 5e^{x/5}$

$V = [5xe^{x/5}]_1^5 - \int_1^5 5e^{x/5}\,dx = (25e - 5e^{1/5}) - [25e^{x/5}]_1^5$

$= 25e - 5e^{1/5} - (25e - 25e^{1/5}) = \boxed{20e^{1/5}}$

Response for question 5(c)



$dV = \pi r^2\,dx$

$= \pi\left(\frac{1}{x^2}\right)^2 dx$

$V = \int_3^\infty \pi\left(\frac{1}{x^2}\right)^2 dx$

$= \pi \lim_{t\to\infty} \int_3^t \frac{1}{x^4}\,dx$

$= \pi \lim_{t\to\infty} \left[\frac{x^{-3}}{-3}\right]_3^t$

$= -\frac{\pi}{3} \lim_{t\to\infty} \left(\frac{1}{t^3} - \frac{1}{27}\right)$

$= \frac{\pi}{3}\left(\frac{1}{27}\right) = \boxed{\frac{\pi}{81}}$

5 5 5 5 5 NO CALCULATOR ALLOWED 5 5 5 5 5

Answer QUESTION 5 part (a) on this page.

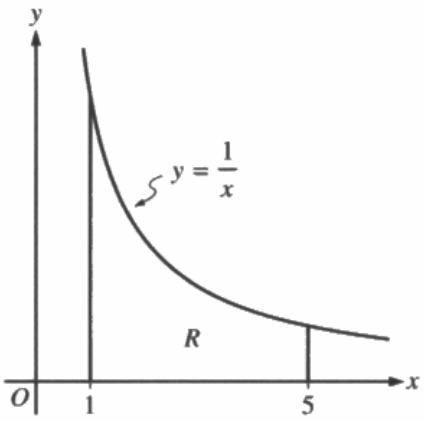


Figure 1

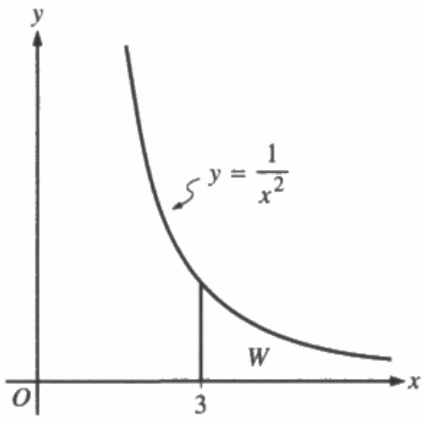


Figure 2

Response for question 5(a)

$\int_1^5 \frac{1}{x}\,dx = \ln x \Big]_1^5$

$= \ln 5 - \ln 1$

$= 0$

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5 5 5 5 5 NO CALCULATOR ALLOWED 5 5 5 5 5

Answer QUESTION 5 parts (b) and (c) on this page.

Response for question 5(b)

$$\int_1^5 xe^{\frac{x}{5}}\,dx$$

$u = x \qquad v = 5e^{\frac{x}{5}}$

$du = dx \qquad dv = e^{\frac{x}{5}}\,dx$

$$5xe^{\frac{x}{5}}\Big]_1^5 - \int_1^5 5e^{\frac{x}{5}}\,dx$$

$$5xe^{\frac{x}{5}}\Big]_1^5 \quad - 25e^{\frac{x}{5}}\Big]_1^5$$

$$\left(5(5)e - 5e^{\frac{1}{5}}\right) - \left(25e - 25e^{\frac{1}{5}}\right)$$

Response for question 5(c)

$$\int_3^\infty \frac{1}{x^2}\,dx$$

$$\lim_{b\to\infty} \int_3^b \frac{1}{x^2}\,dx$$

$$= \lim_{b\to\infty} \left(-\frac{1}{x}\right]_3^b$$

$$= \lim_{b\to\infty} \left(-\frac{1}{b} + \frac{1}{3}\right)$$

$$= \boxed{\frac{1}{3}}$$

Answer QUESTION 5 part (a) on this page.

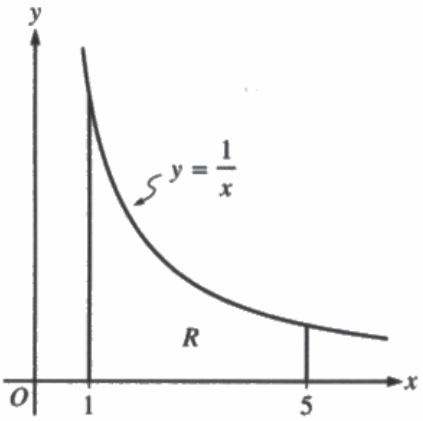

Figure 1

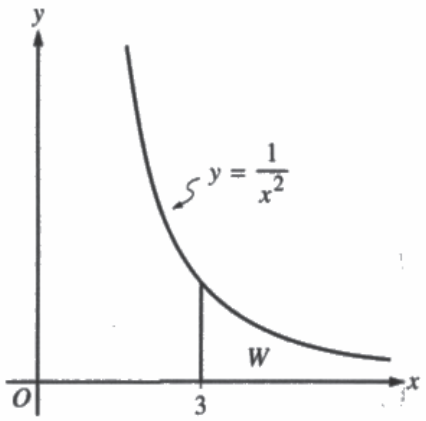

Figure 2

Response for question 5(a)

Area of region R

$\int_1^5 \frac{1}{x}\,dx$ $u = x$ $du = 1$

$= \int_1^5 \frac{du}{u}$

$\ln x \Big|_1^5$

$\ln 5 - \ln 1$

$= \ln 4$

5 5 5 5 5 NO CALCULATOR ALLOWED 5 5 5 5 5

Answer QUESTION 5 parts (b) and (c) on this page.

Response for question 5(b)

$V = \int A$

$A = s^2$

$V = \int_1^5 x e^{x/5}\, dx$

$= \frac{x^2}{5} e^{x/5} \Big|_1^5$

$= \left(\frac{25}{5} e^{5/5}\right) - \left(\frac{1}{5} e^{1/5}\right)$

$= (5e) - \frac{1}{5} e^{1/5}$

Response for question 5(c)

$V = \int_3^0 \left(\frac{1}{x^2} - 0\right)^2 dx$

$V = \int_3^0 \frac{1}{x^4}\, dx$

$4 \ln x \Big|_3^0$

$4 - 4 \ln 3$

Question 5

Note: Student samples are quoted verbatim and may contain spelling and grammatical errors.

Overview

This problem provided two figures, illustrating regions R and W in the first quadrant, associated with the graphs of $y = \frac{1}{x}$ and $y = \frac{1}{x^2}$, respectively.

In part (a) students were asked to find the area of region R. A correct response would recognize the need to compute the definite integral of $y = \frac{1}{x}$ from $x = 1$ to $x = 5$, find an antiderivative of $\ln x$, and present an area of $\ln 5$.

In part (b) students were asked to find the volume of a solid that has region R as its base and whose cross sections perpendicular to the x-axis are rectangles with area $xe^{x/5}$. A correct response would integrate the given area function $y = xe^{x/5}$ from $x = 1$ to $x = 5$ and then proceed to use integration by parts to evaluate the integral.

In part (c) students were asked to find the volume of the solid generated when the unbounded region W is revolved about the x-axis. A correct response would recognize this volume as π times the improper integral of the square of the function $y = \frac{1}{x^2}$, starting at $x = 3$. The response would use correct limit notation to rewrite the improper integral with a variable upper limit, find a correct antiderivative $\left(\int \frac{1}{x^4}\,dx = -\frac{1}{3x^3} + C\right)$, and continue the correct limit notation to find a volume of $\frac{\pi}{81}$.

Sample: 5A
Score: 9

The response earned 9 points: 2 points in part (a), 4 points in part (b), and 3 points in part (c).

In part (a) the response earned the first point with the correct definite integral $\int_1^5 \frac{1}{x}\,dx$ in line 1. The response would have earned the second point with $\ln 5 - \ln 1$ in line 1. In this case, the response correctly simplifies to the boxed answer of $\ln 5$ at the end of line 1 and earned the second point. The response confirms this answer with the sentence in line 2.

In part (b) the response earned the first point with the correct definite integral $\int_1^5 xe^{x/5}\,dx$ in line 2. The response earned the second point by correctly identifying u and dv on line 3. The response correctly applies integration by parts with the expression $\left[5xe^{x/5}\right]_1^5 - \int_1^5 5e^{x/5}\,dx$ in line 5 and earned the third point. The expression at the start of line 6 would have earned the fourth point with no simplification. In this case, simplification to the boxed answer of $20e^{1/5}$ at the end of line 6 is correct, and the fourth point was earned.

Question 5 (continued)

In part (c) the response earned the first point on line 3 with the improper integral $\int_3^{\infty} \pi\left(\frac{1}{x^2}\right)^2 dx$. The response earned the second point with the correct antiderivative $\frac{x^{-3}}{-3}$ in line 5. The response earned the third point with the boxed answer $\frac{\pi}{81}$ at the end of line 7 and the correct use of limit notation in lines 4 through 7.

Sample: 5B
Score: 7

The response earned 7 points: 2 points in part (a), 4 points in part (b), and 1 point in part (c).

In part (a) the response earned the first point with the correct definite integral $\int_1^5 \frac{1}{x}\,dx$ in line 1. The response earned the second point with the circled answer of $\ln 5$ in line 2.

In part (b) the response earned the first point with the correct definite integral $\int_1^5 xe^{\frac{x}{5}}\,dx$ in line 1. The response correctly identifies u and dv in the side work to the right and earned the second point. The response earned the third point with the correct application of integration by parts, $5xe^{\frac{x}{5}}\Big|_1^5 - \int_1^5 5e^{\frac{x}{5}}\,dx,$ in line 2. The response earned the fourth point with the boxed answer in line 4. Numerical simplification is not required.

In part (c) the response did not earn the first point in line 1 with the incorrect improper integral $\int_3^{\infty} \frac{1}{x^2}\,dx$. The response is eligible to earn the second point because the integrand is of the form $\frac{1}{x^n},\ n > 1$. The response earned the second point with $-\frac{1}{x},$ the correct antiderivative of $\frac{1}{x^2}$. The response did not earn the third point because the answer is not correct.

Sample: 5C
Score: 2

The response earned 2 points: 1 point in part (a), 1 point in part (b), and no points in part (c).

In part (a) the response earned the first point with the correct definite integral $\int_1^5 \frac{1}{x}\,dx$ in line 2. The response would have earned the second point with the evaluation $\ln 5 - \ln 1$ in line 5. In this case, there is a simplification error that results in an incorrect boxed answer in line 6. Therefore, the second point was not earned.

In part (b) the response earned the first point with the correct definite integral $V = \int_1^5 xe^{x/5}\,dx$ in line 3. The response did not earn the second, third, and fourth points because the response does not use integration by parts, and the answer is not correct.

Question 5 (continued)

In part (c) the response did not earn the first point because the improper integral in line 1 has limits of integration that are not consistent with region W. The response is eligible to earn the second point because the integrand is of the form $\frac{1}{x^n}$, $n > 1$. The response did not earn the second point because the antiderivative in line 3 is not correct. The response did not earn the third point because the boxed answer is not correct.