2022

AP® | CollegeBoard

AP® Calculus BC

Sample Student Responses and Scoring Commentary

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Free-Response Question 6

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Part B (BC): Graphing calculator not allowed
Question 6 — 9 points

General Scoring Notes
The model solution is presented using standard mathematical notation.

Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be correct to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.

The function f is defined by the power series $f(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots + \frac{(-1)^n x^{2n+1}}{2n+1} + \cdots$ for all real numbers x for which the series converges.

	Model Solution	Scoring	
(a)	Using the ratio test, find the interval of convergence of the power series for f. Justify your answer.		
	$\lim_{n\to\infty}\left\lvert\dfrac{\frac{(-1)^{n+1}x^{2n+3}}{2n+3}}{\frac{(-1)^n x^{2n+1}}{2n+1}}\right\rvert = \lim_{n\to\infty}\left\lvert\dfrac{\frac{x^{2n+3}}{2n+3}}{\frac{x^{2n+1}}{2n+1}}\right\rvert = \lim_{n\to\infty}\left\lvert x^2\left(\frac{2n+1}{2n+3}\right)\right\rvert = \lvert x^2\rvert$	Sets up ratio	**1 point**
	$\lvert x^2\rvert < 1$ for $\lvert x\rvert < 1$. The series converges when $-1 < x < 1$.	Identifies interior of interval of convergence	**1 point**
	When $x = -1$, the series is $-1 + \frac{1}{3} - \frac{1}{5} + \cdots + \frac{(-1)^{n+1}}{2n+1} + \cdots$.	Considers both endpoints	**1 point**
	The series is an alternating series whose terms decrease in absolute value to 0. The series converges by the Alternating Series Test. When $x = 1$, the series is $1 - \frac{1}{3} + \frac{1}{5} + \cdots + \frac{(-1)^n}{2n+1} + \cdots$. The series is an alternating series whose terms decrease in absolute value to 0. The series converges by the Alternating Series Test. Therefore, the interval of convergence is $-1 \le x \le 1$.	Analysis and interval of convergence	**1 point**

Scoring notes:

- A response that includes the substitution error of the form $x^{2(n+1)+1} = x^{2n+3}$ appearing as $x^{2n+1+1} = x^{2n+2}$ in setting up a ratio is eligible for the first 3 points but does not earn the fourth point.
- The first point is earned by presenting a correct ratio with or without absolute values.
- To earn the second point a response must:
  - o use the absolute value of the ratio, or resolve the lack of absolute values by concluding $x^2 < 1$ (without any errors), and correctly evaluate the limit of the ratio, including correct limit notation, and
  - o identify the interior of the interval of convergence. The response can use either interval notation or the compound inequality $-1 < x < 1$ ($|x| < 1$ is insufficient).
- The only incorrect interval eligible for the third point is $0 < x < 1$. In this case, to earn the third point, the response needs to evaluate the general term at $x = 1$.

Total for part (a) 4 points

(b) Show that $\left|f\left(\frac{1}{2}\right) - \frac{1}{2}\right| < \frac{1}{10}$. Justify your answer.

The series for $f\left(\frac{1}{2}\right)$ is an alternating series whose terms decrease in absolute value to 0. The first term of the series for $f\left(\frac{1}{2}\right)$ is $\frac{1}{2}$. Using the alternating series error bound, $f\left(\frac{1}{2}\right)$ differs from $\frac{1}{2}$ by at most the absolute value of the second term of the series. $$\left|f\left(\frac{1}{2}\right) - \frac{1}{2}\right| < \left|\frac{(-1)^1\left(\frac{1}{2}\right)^3}{3}\right| = \frac{1}{24} < \frac{1}{10}$$	Uses second term	**1 point**
	Justification	**1 point**

Scoring notes:

- The first point is earned by correctly using $x = \frac{1}{2}$ in the second term (listing the second term as part of a polynomial is insufficient). Using $x = \frac{1}{2}$ in any term of degree five or higher does not earn this point.
- To earn the second point a response must:
  - o have earned the first point,
  - o state that the series is alternating and that its terms decrease to zero, and
  - o present the inequality $\text{Error} < \frac{1}{24} < \frac{1}{10}$ (or the equivalent).
- A response that states $\text{Error} = \frac{1}{24}$ does not earn the second point.

Total for part (b) 2 points

(c) Write the first four nonzero terms and the general term for an infinite series that represents $f'(x)$.

$f'(x) = 1 - x^2 + x^4 - x^6 + \cdots + (-1)^n x^{2n} + \cdots$	First four terms	**1 point**
	General term	**1 point**

Scoring notes:

- The first point is earned by presenting the first four nonzero terms in a list or as part of a polynomial or series.
- The second point is earned by identifying the general term (either individually or as part of a polynomial or series).
- Read "$=$" as "$\approx$" as necessary.

Total for part (c) **2 points**

(d) Use the result from part (c) to find the value of $f'\left(\frac{1}{6}\right)$.

$f'\left(\frac{1}{6}\right) = 1 - \left(\frac{1}{6}\right)^2 + \left(\frac{1}{6}\right)^4 - \left(\frac{1}{6}\right)^6 + \cdots$ $f'\left(\frac{1}{6}\right)$ is a geometric series with $a = 1$ and $r = -\frac{1}{36}$. $f'\left(\frac{1}{6}\right) = \frac{a}{1-r} = \frac{1}{1-\left(-\frac{1}{36}\right)} = \frac{1}{\frac{37}{36}} = \frac{36}{37}$	Answer	**1 point**

Scoring notes:

- The result from part (c) must be geometric in order to be eligible for this point.
- If a response imports an incorrect geometric series from part (c), this point is earned only for a consistent answer.

Total for part (d) **1 point**

Total for question 6 **9 points**

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Answer QUESTION 6 parts (a) and (b) on this page.

Response for question 6(a)

$$\sum_{n=0} (-1)^n \frac{x^{2n+1}}{2n+1}$$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(-1)^{n+1} x^{2n+3}}{2n+3} \cdot \frac{2n+1}{(-1)^n x^{2n+1}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{x^{2n+3}}{x^{2n+1}} \cdot \frac{2n+1}{2n+3} \right| = \left| x^2 \right| < 1$$

IOC: $-1 \le x \le 1$

ROC = $-1 < x < 1$

@ -1

$$\sum_0^\infty (-1)^n \cdot \frac{(-1)^{2n+1}}{2n+1} = \frac{(-1)^{3n+1}}{2n+1}$$ converges by AST

@ 1

$$\sum_0^\infty (-1)^n \cdot \frac{1^{2n+1}}{2n+1} = \frac{(-1)^n}{2n+1}$$ converges by AST

Response for question 6(b)

$R_n \le |a_{n+2}|$

$R_n \le \left| \frac{x^3}{3} \right|$ @ $x = \frac{1}{2}$

$R_n \le \left| \frac{1}{8} \cdot \frac{1}{3} \right| = \left| \frac{1}{24} \right| < \frac{1}{10}$

The remainder ~~is the~~ less than equal to the next unused term which is the $-\frac{x^3}{3}$ term.

@ $x = \frac{1}{2}$ $\left| -\frac{x^3}{3} \right|$ is $\frac{1}{24}$ which is less than $\frac{1}{10}$

Use a pencil or a pen with black or dark blue ink. Do NOT write your name. Do NOT write outside the box.

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Answer QUESTION 6 parts (c) and (d) on this page.

Response for question 6(c)

$$f'(x) = 1 - x^2 + x^4 - x^6$$

$$= \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

Response for question 6(d)

$$f'\left(\tfrac{1}{6}\right) = 1 - \left(\tfrac{1}{6}\right)^2 + \left(\tfrac{1}{6}\right)^4 - \left(\tfrac{1}{6}\right)^6$$

6 6 6 6 6 NO CALCULATOR ALLOWED 6 6 6 6 6

Answer QUESTION 6 parts (a) and (b) on this page.

Response for question 6(a)

$\frac{(-1)^n x^{2n+1}}{2n+1}$ (absolute value → no $(-1)^n$)

$\lim_{x\to\infty} \left| \frac{x^{2(n+1)+1}}{2(n+1)+1} \cdot \frac{2n+1}{x^{2n+1}} \right| < 1$

$\lim_{x\to\infty} \left| x^2 \cdot \frac{2n+1}{2n+3} \right| < 1$

$|x^2| < 1$

$0 < x^2 < 1$ always >0

$0 < x < 1$

→ test bounds

$x = 1$

$\sum_{n=1}^{\infty} \frac{(-1)^n}{2n+1}$

converges by alternating series test

$x = 0$

$\sum_{n=1}^{\infty} 0$ → converges to 0

$\boxed{[0, 1]}$

Response for question 6(b)

$|f(\frac{1}{2}) - \frac{1}{2}| < \frac{1}{10}$

$|f(\frac{1}{2}) - \frac{1}{2}| < a_{n+1}$ (or next term)

$a_{n+1} = \frac{x^3}{3}$

$a_{n+1} = \frac{(\frac{1}{2})^3}{3}$

$a_{n+1} = \frac{1}{8(3)} = \frac{1}{24} < \frac{1}{10}$ ✓

Use a pencil or a pen with black or dark blue ink. Do NOT write your name. Do NOT write outside the box.

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6 6 6 6 6 NO CALCULATOR ALLOWED 6 6 6 6 6

Answer QUESTION 6 parts (c) and (d) on this page.

Response for question 6(c)

$$f'(x) = 1 - x^2 + x^5 - x^7 + \dots + \frac{(-1)^n (2n+1) x^{2n}}{(2n+1)} + \dots$$

$$f'(x) = 1 - x^2 + x^5 - x^7 + \dots + (-1)^n x^{2n} + \dots$$

Response for question 6(d)

$$f'\left(\tfrac{1}{6}\right) \approx 1 - \left(\tfrac{1}{6}\right)^2 + \left(\tfrac{1}{6}\right)^5 - \left(\tfrac{1}{6}\right)^7$$

$$f'\left(\tfrac{1}{6}\right) \approx 1 - \frac{1}{36} + \frac{1}{6^5} - \frac{1}{6^7}$$

Use a pencil or a pen with black or dark blue ink. Do NOT write your name. Do NOT write outside the box.

6 6 6 6 6 **NO CALCULATOR ALLOWED** 6 6 6 6 6

Answer QUESTION 6 parts (a) and (b) on this page.

Response for question 6(a)

① Ratio Test

$$\sum \frac{(-1)^n x^{2n+1}}{(2n+1)}$$

$$\left| \frac{x^{(2(n+1)+1)}}{(2(n+1)+1)} \cdot \frac{2n+1}{x^{2n+1}} \right| < 1$$

$$\left| \frac{x^{\cancel{2n}+3}}{\cancel{2n+2}} \cdot \frac{\cancel{(2n+1)}}{\cancel{x^{2n+1}}} \right| < 1$$

$$\left| \frac{x^{2n+3}}{x^{2n+1}} \right| < 1$$

$|x^2| < 1$ ROC

$-1 < x < 1$

② Test pts

$x = -1$

$$\sum \frac{(-1)^{-1}(x)^{2(-1)+1}}{(2(-1)+1)} = \frac{1}{(-1)(x^{-})(2(-1)+1)} \sum \frac{1}{x}$$

so div (comp to $\sum \frac{1}{n}$ harmonic p series)

$x = 1$

$$\sum \frac{(-1)^1(x)^{2(1)+1}}{(2(1)+1)} = x \ldots$$

goes to ∞

$\boxed{-1 < x < 1}$

Response for question 6(b)

Lagrange

$$\frac{f^{n+1}(\max)}{(n+1)!} (x+c)^{n+1}$$

$$\frac{f^{(8)}(.8)}{8!} (x - \tfrac{1}{2})^8 < \frac{1}{10}$$

$$\frac{\left(\frac{1}{2}\right)^9}{9} < \frac{1}{10}$$

$f^{(8)}$

$$f^8(8) = \frac{(-1)^8 (x^{2(8)+1})}{2(8)+1}$$

$$\frac{\cancel{(-1)^8} (x^{17})}{17}$$

$$f^{(4)} = \frac{\cancel{(-1)^4} (x^{2(4)+1})}{2(4)+1} = \boxed{\frac{x^9}{9}}$$

max error

4th term

6 6 6 6 6 NO CALCULATOR ALLOWED 6 6 6 6 6

Answer QUESTION 6 parts (c) and (d) on this page.

Response for question 6(c)

Find f'

$f = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots + \frac{(-1)^n x^{2n+1}}{2n+1}$

$f' = 1 - \frac{\cancel{3}x^2}{\cancel{3}} + \frac{\cancel{5}x^4}{\cancel{5}} - \frac{\cancel{7}x^6}{\cancel{7}}$

$f' = 1 - x^2 + x^4 - x^6 + \dots + \frac{n(-1)^{n-1} x^{(2n)}}{(2n)}$

Response for question 6(d)

$f'(\frac{1}{6}) = 1 - (\frac{1}{6})^2 + (\frac{1}{6})^4 - (\frac{1}{6})^6$

Use a pencil or a pen with black or dark blue ink. Do NOT write your name. Do NOT write outside the box.

Question 6

Note: Student samples are quoted verbatim and may contain spelling and grammatical errors.

Overview

In this problem a function f is defined by a power series for all real numbers for which the power series converges.

In part (a) students were asked to use the ratio test to find the interval of convergence for the power series. A correct response should set up the ratio $\left|\frac{a_{n+1}}{a_n}\right|$ and find the limit of this ratio as $n \to \infty$ to find the interior of the interval of convergence. The response should then use the Alternating Series Test to determine that the series converges for both endpoints of this interval of convergence.

In part (b) students were asked to justify that $\left|f\left(\frac{1}{2}\right) - \frac{1}{2}\right| < \frac{1}{10}$. A correct response would recognize that $\frac{1}{2}$ is the first term of the series for $f\left(\frac{1}{2}\right)$ and that the series for f is alternating with terms that decrease in absolute value to 0. Therefore, $\left|f\left(\frac{1}{2}\right) - \frac{1}{2}\right|$ must be no more than the second term of the series for $f\left(\frac{1}{2}\right)$, which is $\frac{1}{24} < \frac{1}{10}$.

In part (c) students were asked to write the first four terms and the general term for an infinite series that represents $f'(x)$. A correct response would differentiate the first four given terms of the series for $f(x)$.

Finally, in part (d) students were asked to use the series from part (c) to find the value of $f'\left(\frac{1}{6}\right)$. A correct response must recognize that the series for $f'\left(\frac{1}{6}\right)$ is geometric with $a = 1$ and $r = -\frac{1}{36}$. The value of $f'\left(\frac{1}{6}\right)$ is the sum of the geometric series, $\frac{36}{37}$.

Sample: 6A
Score: 7

The response earned 7 points: 4 points in part (a), 1 point in part (b), 2 points in part (c), and no points in part (d).

In part (a) the response earned the first point by presenting the correct ratio. The response earned the second point by presenting the correct interior of the interval of convergence with the correct limit notation. The response earned the third point by evaluating the general term at $x = -1$ and $x = 1$. The response earned the fourth point by stating "converges by AST" and presenting the correct interval of convergence.

In part (b) the response earned the first point in the third line by using the second term of the series evaluated at $\frac{1}{2}$. The response did not earn the second point because there is no reference to this as an alternating series whose terms decrease to 0.

In part (c) the response earned both the first point and the second point by presenting the correct terms.

In part (d) the response does not present the correct answer and did not earn the point.

Question 6 (continued)

Sample: 6B
Score: 4

The response earned 4 points: 2 points in part (a), 1 point in part (b), 1 point in part (c), and no points in part (d).

In part (a) the response earned the first point in the second line by presenting the correct ratio. The response did not earn the second point because an incorrect interval is presented. The response presents the only other eligible interval of $0 < x < 1$ and, therefore, is eligible for the third point. The response earned the third point by evaluating the general term at $x = 1$. The response did not earn the fourth point because the interval of convergence is incorrect.

In part (b) the response earned the first point by using the second term evaluated at $\frac{1}{2}$. The response did not earn the second point because there is no reference to this as an alternating series whose terms decrease to 0.

In part (c) the response did not earn the first point because the terms are not all correct. The response earned the second point because the general term is correct.

In part (d) the response does not present the correct answer and did not earn the point.

Sample: 6C
Score: 2

The response earned 2 points: 1 point in part (a), no points in part (b), 1 point in part (c), and no points in part (d).

In part (a) the response earned the first point by presenting the correct ratio. The response did not earn the second point because there is no limit notation. The response did not earn the third point because the general term is incorrectly evaluated at both $x = 1$ and $x = -1$. The response did not earn the fourth point because the interval of convergence is incorrect.

In part (b) the response did not earn the first point because the incorrect term is used. The response is not eligible for the second point.

In part (c) the response earned the first point by presenting the correct first four nonzero terms. The response did not earn the second point because the general term is incorrect.

In part (d) the response does not present the correct answer and did not earn the point.