Chief Reader Report on Student Responses:
2023 AP® Calculus AB/BC Free-Response Questions

Number of Readers (Calculus AB/Calculus BC): 1,359

Calculus AB

- Number of Students Scored: 273,987
- Score Distribution

Exam Score	N	%At
5	61,350	22.39
4	44,328	16.18
3	53,162	19.40
2	59,503	21.72
1	55,644	20.31

- Global Mean: 2.99

Calculus BC

- Number of Students Scored: 135,458
- Score Distribution

Exam Score	N	%At
5	58,991	43.55
4	21,478	15.86
3	25,803	19.05
2	20,590	15.20
1	8,596	6.35

- Global Mean: 3.75

Calculus BC Calculus AB Subscore

- Number of Students Scored: 135,458
- Score Distribution

Exam Score	N	%At
5	62,847	46.40
4	26,538	19.59
3	25,463	18.80
2	11,646	8.60
1	8,964	6.62

- Global Mean: 3.91

* The number of students with Calculus AB subscores may differ slightly from the number of students who took the AP Calculus BC Exam due to exam administration incidents.

The following comments on the 2023 free-response questions for AP® Calculus AB and Calculus BC were written by the Chief Reader, Julie Clark of Hollins University. They give an overview of each free-response question and of how students performed on the question, including typical student errors. General comments regarding the skills and content that students frequently have the most problems with are included. Some suggestions for improving student preparation in these areas are also provided. Teachers are encouraged to attend a College Board workshop to learn strategies for improving student performance in specific areas.

Question AB/BC1

Topic: Modeling Rates with Riemann Sum and MVT
Max Score: 9
Mean Score: AB1 3.79
Mean Score: BC1 5.26

What were the responses to this question expected to demonstrate?

In this problem students were given a table of times t in seconds and values of a function $f(t)$, which models the rate of flow of gallons of gasoline pumped into a gas tank.

In part (a) students were asked to interpret the meaning of $\int_{60}^{135} f(t)\,dt$ using correct units. Then students were asked to use a right Riemann sum with three subintervals to approximate the value of this integral. A correct response will indicate that the integral represents the accumulated gallons of gasoline pumped into the tank during the time interval from $t = 60$ to time $t = 135$ seconds. The approximation is found using the following expression:
$(90 - 60) \cdot f(90) + (120 - 90) \cdot f(120) + (135 - 120) \cdot f(135)$.

In part (b) students were asked to justify whether there must be a value of c, with $60 < c < 120$, such that $f'(c) = 0$. Students are expected to note that because the function f is known to be differentiable on the interval $(0, 150)$, it must be continuous on the subinterval $[60, 120]$. Therefore, because the average rate of change of f on the interval $[60, 120]$ is equal to 0, such a value of c is guaranteed by the Mean Value Theorem.

In part (c) the function $g(t) = \left(\frac{t}{500}\right)\cos\left(\left(\frac{t}{120}\right)^2\right)$ was introduced as a second function that modeled the rate of flow of the gasoline. Students were asked to use the model g to find the average rate of flow of the gasoline over the time interval $0 \le t \le 150$. A correct response will show the setup $\frac{1}{150 - 0} \cdot \int_0^{150} g(t)\,dt$ and then use a calculator to find the value 0.096 gallon per second.

In part (d) students were asked to find the value of $g'(140)$ and interpret the meaning of this value in the context of the problem. A correct response will use a calculator to find $g'(140) = -0.005$ and report that at time $t = 140$ seconds the rate at which gasoline is flowing into the tank is decreasing at a rate of 0.005 gallon per second per second.

How well did the responses address the course content related to this question? How well did the responses integrate the skills required on this question?

<u>Part (a)</u>

- Most responses were able to correctly interpret the meaning of this integral.
- Most of the responses also gave correct units. The few errors seen in providing the units were typically a failure to include "gallons" or a reference to time.
- Responses were largely successful in approximating the value of $\int_{60}^{135} f(t)\,dt$ using a right Riemann sum. Only a few responses had errors in setting up the approximation—for example, some used a left Riemann sum instead.

Part (b)

- A large proportion of the responses knew that applying a theorem was the intent of this question. The most commonly referenced theorems were the Mean Value Theorem and Rolle's Theorem.
- Many responses earned the first point by demonstrating that $f(120) - f(60) = 0$ or that $f(120) = f(60)$.
- Quite a few responses failed to explain that the differentiability of the function f allowed the conclusion that f is continuous on the required closed interval.
- A few responses used the Extreme Value Theorem to argue that $f(90) = 0.15$ is the absolute maximum value of f on $[60, 120]$. However, this argument is not valid. In addition, knowing the maximum value of f does not address the question asked.

Part (c)

- Almost all responses were able to earn some points in this part.
- Most responses correctly provided or used the average value formula, $\frac{1}{150-0}\int_0^{150} g(t)\,dt$.
- Most of these responses went on to use a calculator to evaluate this average value correctly.

Part (d)

- A majority of the responses were able to correctly find the value of $g'(140)$ using a calculator.
- A small number of responses successfully differentiated g by hand using the product and chain rules and then used a calculator to correctly evaluate this derivative at $t = 140$.
- A large proportion of the responses struggled to correctly interpret this derivative, although a few were successful.

What common student misconceptions or gaps in knowledge were seen in the responses to this question?

Common Misconceptions/Knowledge Gaps	*Responses that Demonstrate Understanding*
• In part (a) some responses interpreted the given integral as a rate, e.g., gallons per second. • A few responses provided a left Riemann sum rather than a right Riemann sum.	• $\int_{60}^{135} f(t)\,dt$ is the total gallons of gasoline pumped into the tank during the time interval from $t = 60$ to time $t = 135$ seconds. • $\int_{60}^{135} f(t)\,dt \approx (0.15)(30) + (0.1)(30) + (0.05)(15)$
• In part (b) many responses failed to state that f is continuous because f is differentiable, including many just claiming that f is continuous with no justification. • Some responses attempted to reference the Intermediate Value Theorem by comparing the slopes of the graph of f on the intervals $[60, 90]$ and $[90, 120]$.	• f is differentiable on $[60, 120] \Rightarrow f$ is continuous on $[60, 120]$. • $\frac{f(120) - f(60)}{120 - 60} = \frac{0.1 - 0.1}{60} = 0$ Therefore, by the Mean Value Theorem, there must exist a c, with $60 < c < 120$, such that $f'(c) = 0$.

• In part (c) many responses demonstrated an incomplete understanding of the concept of the "average rate of flow." These responses tried to sum various values of $g(t)$ and then divided the sum by 150. • Some responses failed to divide the value of $\int_0^{150} g(t)\,dt$ by 150.	• The average rate of flow over the time interval $0 \le t \le 150$ is $\frac{1}{150-0} \cdot \int_0^{150} g(t)\,dt = 0.096.$
• In part (d) many responses attempted to differentiate $g(t)$ by hand, indicating a lack of familiarity with using a calculator to find $g'(140)$. • Many responses provided an incorrect interpretation of $g'(140)$, failing to describe this value as a rate of a rate or failing to indicate that the rate of the rate was changing (or decreasing).	• $g'(140) = -0.005$ • The exact value of $g'(140)$ is $\frac{1}{500}\cos\left(\frac{49}{36}\right) - \frac{49}{9000}\sin\left(\frac{49}{36}\right).$ • The rate at which gasoline is flowing into the tanks is decreasing at a rate of 0.005 gallon per second per second at time $t = 140$ seconds.

Based on your experience at the AP® Reading with student responses, what advice would you offer teachers to help them improve the student performance on the exam?

- Responses indicate that most students are familiar with theorems commonly used in calculus but are unclear as to how they differ and how mathematicians use them to justify answers. Teachers could help students develop a deeper understanding of the underpinnings of calculus by focusing on understanding and correctly stating the required initial conditions of each theorem. This will also help students develop the important skill of writing strong justifications.
- These responses also indicate that students are quite capable of using calculators to find the values of definite integrals and derivatives at a point. However, students frequently do not fully understand the meaning of said values. An understanding of how derivatives and integrals create meaning in context is an important skill for students to develop. Being able to interpret meaning from existing units and adjust units after integrating or differentiating allows students to apply calculus to real-world problems and understand why calculus is an important part of mathematical study.

What resources would you recommend to teachers to better prepare their students for the content and skill(s) required on this question?

- The content and skills required on this question are found in Topic 2.3 (Estimating Derivatives of a Function at a Point), Topic 5.1 (Using the Mean Value Theorem), Topic 6.2 (Approximating Areas with Riemann Sums), Topic 8.1 (Finding the Average Value of a Function on an Interval), and Topic 8.3 (Using Accumulation Functions and Definite Integrals in Applied Contexts). The AP Daily videos for each of these topics are excellent resources for developing understanding of required content and skills.
- Skill 3.F (Explain the meaning of mathematical solutions in context) was key in parts (a) and (d) of this question. The course framework (CF) and the question itself are important resources to help students prepare to write complete interpretations and explanations. When learning new material (e.g., Topic 8.3) teachers can guide students to use language in the CF as a template:

  > EK CHA-4.D.2: [*The definite integral of the rate of change of a quantity over an interval*] gives the net change of [*that quantity over that interval*].

  With practice using a variety of questions available in the question bank, students will be prepared to apply this technique to write a complete interpretation in part (a) of this question:

$\left[\int_{60}^{135} f(t)\,dt\right]$ gives the net change of [gallons of gasoline pumped over the interval $60 \le t \le 135$ seconds].

Note that the response starts from the form of the EK and includes specific language included in the question. This approach utilizes variations of two instructional strategies found in the Course and Exam Description (CED): Model Questions (page 208) and Sentence Starters (page 212).

- Students should practice all year communicating reasoning, showing work, and presenting setups for work performed in a calculator:
  - Part (a) of this question requires that students show sufficient work to demonstrate the use of a right Riemann sum, as described in the scoring notes for the second point.
  - Part (b) requires that students communicate the Mean Value Theorem (MVT) connection between $f'(c)$ and the average rate of change and that they demonstrate that the MVT applies by reasoning that *because* f is differentiable, f must also be continuous (a condition for application of the MVT—see skill 3.C).
  - Part (c) requires that students present the average value setup (which might be done in one or more steps) and the correct answer rounded appropriately (see skill 4.E). Using precise mathematical language (skill 4.A) is critical, especially if a student attempts to split this work by evaluating the definite integral and then dividing by 150.
  - Part (d) also requires attention to precise mathematical language (see skill 4.A) if a student interprets the sign of $g'(140)$.

One resource to support the development of communication skills is the formative topic questions and progress checks provided with each topic or unit on AP Classroom, especially the free-response questions. Giving students regular feedback on their work using the scoring guidelines provided with each formative assessment free-response question is an excellent way to help students to develop these skills over time.

Question AB2

Topic: Particle Motion—Acceleration-Displacement-Distance
Max Score: 9
Mean Score: 4.62

What were the responses to this question expected to demonstrate?

In this problem students were told that Stephen is swimming back and forth along a straight path in a 50-meter pool for 90 seconds with a velocity modeled by the function $v(t) = 2.38e^{-0.02t} \sin\left(\frac{\pi}{56}t\right)$. Here t is measured in seconds and $v(t)$ is measured in meters per second.

In part (a) students were asked to find all times t in the interval $0 < t < 90$ at which Stephen changes direction. A correct response will indicate understanding that Stephen changes direction when his velocity changes sign. Based on the model for velocity, it is clear that $v(56) = 0,$ and a calculator could be used to confirm that $v(t)$ does change sign at this time.

In part (b) students were asked to find Stephen's acceleration at time $t = 60$ seconds and to indicate units of measure. Then students were asked to provide a reason for whether Stephen was speeding up or slowing down at this time. A correct response will communicate that Stephen's acceleration is the derivative of his velocity and will use a calculator to find $a(60) = v'(60) = -0.036$ meter per second per second. Then the response will evaluate the given velocity function at $t = 60$ in order to determine that $v(60)$ is also negative and conclude that, because the velocity and acceleration have the same sign at $t = 60,$ Stephen must be speeding up.

In part (c) students were asked to find the distance between Stephen's position at time $t = 20$ seconds and his position at time $t = 80$ seconds. A correct response will show the calculator setup $\int_{20}^{80} v(t)\, dt,$ with a numerical answer of 23.384 meters.

In part (d) students were asked to find the total distance Stephen swims over the time interval from $t = 0$ to $t = 90$ seconds. A correct response will provide the setup $\int_{0}^{90} |v(t)|\, dt$ and use a calculator to find the total distance is 62.164 meters.

How well did the responses address the course content related to this question? How well did the responses integrate the skills required on this question?

Part (a)

- A majority of the responses considered the sign of Stephen's velocity in an attempt to determine when Stephen's direction changed, and most of these responses correctly reported time $t = 56.$
- Some responses failed to reason that Stephen changed direction because his velocity changed signs, instead stating only that $v(56) = 0.$
- Some responses correctly reasoned that Stephen changed direction at time $t = 56$ because $v(56) = 0$ and $a(56) = v'(56) \neq 0,$ which is equivalent to the Second Derivative Test.

Part (b)

- Most responses used a calculator to find the correct value of $a(60)$. However, many responses did not provide the setup for the response, i.e., $a(60)$ is $v'(60)$.
- Several responses did not report any units for the acceleration, but most responses provided the correct units of meters per second per second.
- A large proportion of the responses presented the misconception that the value of $a(60)$ alone could be used to determine whether Stephen was speeding up or slowing down.
- The responses that considered the values $a(60)$ and $v(60)$ were generally successful in reasoning that Stephen was speeding up.

Part (c)

- Most responses indicated an understanding that the distance between Stephen's position at time $t = 20$ and time $t = 80$ was found by evaluating $\int_{20}^{80} v(t)\,dt$, although some responses presented this as $\int_{0}^{80} v(t)\,dt - \int_{0}^{20} v(t)\,dt$.
- Surprisingly, a number of responses that provided the correct integral setup did not evaluate the integral correctly using their calculator, although some of the errors were just a failure to present answers accurate to three digits after the decimal point.

Part (d)

- About half of the responses presented a correct integral expression for the total distance Stephen swam over the time interval $0 \le t \le 90$. The most common correct setups were $\int_{0}^{90} |v(t)|\,dt$ and $\int_{0}^{56} v(t)\,dt + \left|\int_{56}^{90} v(t)\,dt\right|$.
- Quite a few responses presented a setup of $\int_{0}^{90} v(t)\,dt$, or an equivalent expression involving multiple integrals, demonstrating an incomplete understanding of the difference between displacement and total distance traveled.

What common student misconceptions or gaps in knowledge were seen in the responses to this question?

Common Misconceptions/Knowledge Gaps	*Responses that Demonstrate Understanding*
• In part (a) many responses reasoned that Stephen changes direction at $t = 56$ because $v(56) = 0$. • Some responses made ambiguous statements such as "at $t = 56$ it changes sign."	• Stephen changes direction at $t = 56$ because $v(t)$ changes sign there.
• In part (b) some responses failed to provide the setup used on their calculator, e.g., $a(60) = -0.036$. • Some responses presented no or incorrect acceleration units, e.g., meters per second. • Many responses incorrectly reasoned that Stephen was slowing down because $a(60) < 0$.	• $a(60) = v'(6) = -0.036$ • $a(60) = v'(6) = -0.036$ meter per second per second • $a(60) = -0.036$, $v(60) = -0.160 < 0$ Because $a(60)$ and $v(60)$ have the same sign, Stephen is speeding up.

• Quite a few responses presented poor notation and/or communication such as $a(t) = v'(t) = -0.036$, $\frac{dy}{dx}[v(t)]_{t=60} = -0.036$, or $\frac{d}{dx}[v(60)]_{t=60} = -0.036$.	• $a(t) = v'(t),\ a(60) = -0.036$ $\frac{d}{dt}[v(t)]_{t=60} = -0.036$
• In part (c) some responses confused displacement with total distance traveled, e.g., $\int_{20}^{80} \lvert v(t) \rvert\, dt = 39.282$ or $\int_{20}^{56} v(t)\, dt - \int_{56}^{80} v(t)\, dt$. • A few responses presented an unreasonable answer, e.g., distance $= \int_{0}^{20} v(t)\, dt - \int_{0}^{80} v(t)\, dt = -23.384$.	• Distance $= \int_{20}^{80} v(t)\, dt = 23.384$
• In part (d), as in part (c), some responses confused displacement with total distance traveled, e.g., $\int_{0}^{90} v(t)\, dt = 37.679$ or $\left\lvert \int_{0}^{90} v(t)\, dt \right\rvert = 37.679$.	• Total distance $= \int_{0}^{90} \lvert v(t) \rvert\, dt = 62.164$

Based on your experience at the AP® Reading with student responses, what advice would you offer teachers to help them improve the student performance on the exam?

- Teachers could provide students with multiple opportunities to connect the concepts of displacement, total distance traveled and position—pointing out the similarities between these concepts and how various definite integrals involving velocity and speed correspond to these values.
- In addition, teachers could help students see the connection between a definite integral that measures net change (without context) and the definite integral for displacement in the context of particle motion. Similarly, students should see the connection between a definite integral representing the total accumulation (without context) and the definite integral that measures total distance traveled in the context of particle motion.
- Proper notation is critical for success as it reinforces the meaning of the concepts in the course and can support student understanding in calculus. Teachers can help students develop a command of proper notation by consistently modeling proper notation and implementing a variety of notational approaches throughout the course. Additionally, teachers can hold students accountable for their own notation consistently throughout the course.
- Teachers could frequently reinforce the importance and necessity of supporting work in mathematics as a method of communication. Teachers could present students with exemplary responses that display clear and concise work as models for free-response questions throughout the year to help students develop their communication and reasoning skills.
- On calculator-active questions, many students are unaware that they are expected to communicate how they are utilizing their graphing calculator by showing the necessary setup prior to presenting numerical answers. This setup requires proper notation, and responses that lack notation or include incorrect notation are rarely eligible to earn all possible points. Teachers could require similar setups on all submitted work in their classrooms to ensure that this becomes reflexive for students.
- Students need guidance and support as they develop an understanding of what qualifies as proper mathematical reasoning. Teachers can reinforce these ideas by requiring students to use the concepts of calculus to explain their answers and within their reasoning. Providing students frequent opportunities to discuss calculus, explain answers, and provide mathematical reasoning throughout the year is an essential step to promote and strengthen these skills.

What resources would you recommend to teachers to better prepare their students for the content and skill(s) required on this question?

- This question is a classic particle motion problem applied to the context of Stephen swimming back and forth along a straight path in a pool.
- Content: Parts (a) and (b) assess understanding of particle motion in differential calculus (Topic 4.2—Straight-Line Motion: Connecting Position, Velocity, and Acceleration). Parts (c) and (d) assess understanding of application of integration to a particle motion problem (Topic 8.2—Connecting Position, Velocity, and Acceleration Functions Using Integrals). AP Daily videos for these topics are very helpful resources for developing understanding of these concepts.
- Skills: After identifying the key underlying structure of particle motion in this context (skill 2.A), this question became an exercise in identifying (skill 1.D) and applying (skill 1.E) appropriate mathematical rules or procedures, with and without technology. It is important to require students to show setups for work done using a calculator and to emphasize all year long that students' calculators should be in radian mode for calculus.

  Past AP exam questions and formative assessment questions on AP Classroom (topic questions and progress checks—both multiple-choice questions and free-response questions) are excellent resources for student practice. Giving students consistent, high-quality feedback using resources provided with formative assessment questions on AP Classroom can help them to master these key skills and best practices.

Question AB3/BC3

Topic: Modeling with Differential Equation
Max Score: 9
Mean Score: AB3 2.07
Mean Score: BC3 3.74

What were the responses to this question expected to demonstrate?

In this problem students were told that an increasing function M models the temperature of a bottle of milk taken out of the refrigerator and placed in a pan of hot water to warm. The function M satisfies the differential equation $\frac{dM}{dt} = \frac{1}{4}(40 - M),$ where t is measured in minutes since the bottle was placed in the pan and M is measured in degrees Celsius. At time $t = 0,$ the temperature of the milk is $5°C.$

In part (a) students were given a slope field for the differential equation and asked to sketch the solution curve through the point $(0, 5).$ A correct response will draw a curve that passes through the point $(0, 5),$ follows the indicated slope segments, extends reasonably close to the left and right edges of the slope field, and lies entirely below the horizontal line segments at $M = 40.$

In part (b) students were asked to use the line tangent to the graph of M at $t = 0$ to approximate $M(2).$ A correct response will find the slope of the tangent line when $t = 0$ is $\left.\frac{dM}{dt}\right|_{t=0} = \frac{35}{4}$ and then use the tangent line equation, $y = 5 + \frac{35}{4}t,$ to find that $M(2) \approx 22.5.$

In part (c) students were asked to find an expression for $\frac{d^2M}{dt^2}$ in terms of M and then to use $\frac{d^2M}{dt^2}$ to reason whether the approximation from part (b) is an underestimate or overestimate for the actual value of $M(2).$ A correct response will differentiate the given differential equation to obtain $\frac{d^2M}{dt^2} = -\frac{1}{4}\frac{dM}{dt} = -\frac{1}{16}(40 - M),$ then use the information that $M(t) < 40$ to determine that the second derivative of M is negative and therefore the graph of M is concave up and the approximation in part (b) is an overestimate.

In part (d) students were asked to use separation of variables to find an expression for the particular solution to the given differential equation with initial condition $M(0) = 5.$ A correct response will separate the variables, integrate, use the initial condition to find the constant of integration, and arrive at a solution of $M = 40 - 35e^{-t/4}.$

How well did the responses address the course content related to this question? How well did the responses integrate the skills required on this question?

Part (a)

- Most responses sketched a solution curve that began at the point $(0, 5)$ and stayed within the given slope field.
- Some responses failed to demonstrate the correct asymptotic behavior of the solution curve and some failed to extend the curve close to the right edge of the slope field.
- Some solution curves appeared to be constant along $M = 40$, or crossed above $M = 40$.

Part (b)

- Many responses demonstrated an understanding of needing to use $\frac{dM}{dt}$ to find the slope of the tangent line, and many responses used a correct equation of the tangent line to find a correct approximation.
- Some responses used Euler's method to find the approximation and because, with one step, this is equivalent to a tangent line approximation, these responses earned both points.
- A few responses found the expression, $M - 5 = \frac{35}{4} \cdot 2$, but then failed to solve for M, thereby never presenting an approximation.

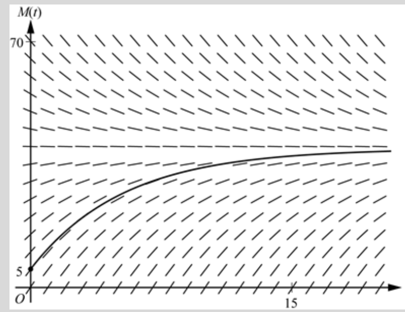
Part (c)

- Responses that correctly used the chain rule were able to find the correct second derivative, $\frac{d^2M}{dt^2}$. However, some of these responses did not express this derivative in terms of M, instead leaving $\frac{d^2M}{dt^2}$ as $-\frac{1}{4}\frac{dM}{dt}$.
- Many responses demonstrated an understanding that the sign of the second derivative, or a decreasing first derivative, or the fact that M is concave down could be used to reason that the approximation is an overestimate.
- Most responses had more difficulty explaining their reasoning than calculating the second derivative.

Part (d)

- Responses generally indicated an understanding of the steps required to find the particular solution, but few were able to successfully complete all of the required processes.
- A significant number of responses correctly separated the variables in a variety of ways, found acceptable antiderivatives, and correctly used the initial condition with an appropriate constant of integration.
- However, many responses reported $\int \frac{dM}{40 - M} = \ln(40 - M)$, rather than $\int \frac{dM}{40 - M} = -\ln(40 - M)$.
- Quite a few responses were unable to correctly simplify logarithmic or exponential expressions.

What common student misconceptions or gaps in knowledge were seen in the responses to this question?

Common Misconceptions/Knowledge Gaps	*Responses that Demonstrate Understanding*
• In part (a) some responses presented solution curves that lay above and below the horizontal asymptote $M = 40$. • A few responses presented a solution curve through the point $(5, 0)$ rather than $(0, 5)$.	• 
• In part (b) some responses used the value of $M = 0$ rather than $M = 5$ to find the slope of the tangent line, e.g., $\left.\frac{dM}{dt}\right\|_{t=5} = \frac{1}{4}(40-0) = 10$. • Some responses found the tangent line at $t = 2$ instead of the tangent at $t = 0$. • Some responses illustrated a significant misconception of a first-order differential equation by attempting to solve the differential equation for $M(t)$ in this part.	• $\left.\frac{dM}{dt}\right\|_{t=0} = \frac{1}{4}(40-5) = \frac{35}{4}$ • The tangent line equation is $y = 5 + \frac{35}{4}(t-0)$. • $M(2) \approx 5 + \frac{35}{4} \cdot 2 = 22.5$
• In part (c) many responses did not use the chain rule to find $\frac{d^2M}{dt^2}$, resulting in a response of $\frac{d^2M}{dt^2} = -\frac{1}{4}$. • Other responses used vague language such as "because it is negative" or "the function is concave down," making it impossible to know to what function a response was referring.	• $\frac{d^2M}{dt^2} = -\frac{1}{4}\frac{dM}{dt} = -\frac{1}{4}\left(\frac{1}{4}(40-M)\right) = -\frac{1}{16}(40-M)$ • Therefore, the tangent line approximation of $M(2)$ is an overestimate.
• In part (d) a handful of responses did not know how to separate the variables; others did not separate correctly. • Many responses demonstrated incorrect simplifications, e.g., $e^{t/4+C} = e^{t/4} + e^C$ or $-\ln(40-M) = \frac{1}{4}t + C \Rightarrow \frac{1}{40-M} = e^{t/4} + e^C$.	• $\frac{dM}{40-M} = \frac{1}{4}\,dt$ • $-\ln\|40-M\| = \frac{1}{4}t + C$ $-\ln\|40-5\| = 0 + C \Rightarrow C = -\ln 35$ $\ln(40-M) = -\frac{1}{4}t + \ln 35$ $40 - M = 35e^{-t/4} \Rightarrow M = 40 - 35e^{-t/4}$

Based on your experience at the AP® Reading with student responses, what advice would you offer teachers to help them improve the student performance on the exam?

- Teachers could emphasize how information given in the question can be used to create a mental picture of the solution curve. In this case, because the prompt stated that M is increasing and $M(t) < 40$ for all values of t, a solution curve that passes through $(0, 5)$ must have a horizontal asymptote. Teachers could provide students with practice sketching solution curves of various types of solutions (transient, steady-state, periodic, unbounded).

- Teachers could begin the exploration of each differential equation by establishing the role of each variable in the equation. Then remind students that the derivative tells us the slope of a differentiable curve and that the derivative may depend on the independent variable, the dependent variable, or both.
- Teachers could provide practice, practice, practice finding and using the equation of a tangent line.
- Teachers could never provide enough practice with manipulating logarithmic or exponential equations or equations involving absolute values.
- Teachers could provide several options (correct and incorrect) of “separated” separable differential equations and have students practice classifying the options as “correctly separated,” “poorly separated,” or “not separated.”

What resources would you recommend to teachers to better prepare their students for the content and skill(s) required on this question?

Both AB and BC students found this to be the most challenging question on their exams. In each part of the question, students missed opportunities to earn points:

- In part (a) students seemed to be entering the graphing of the solution curve with a baseline understanding of the task, but some did not use appropriate graphing techniques (skill 4.D). For example, some were not careful to make sure that their solution curve passed through the point $(0, 5)$ or that the curve approached the horizontal segments at $M = 40$ while staying below the implied line at $M = 40.$ AP Daily videos for Topic 7.3 clearly describe and model good technique. Instructional strategies to practice and develop appropriate graphing techniques can be found on page 220 of the Course and Exam Description (CED). In addition to these, the instructional strategy “Error Analysis” (page 206 of the CED) would be a useful resource to help students develop strong graphing techniques.
- In part (b) a tangent line approximation should be quite straightforward. However, students needed to recall that they can use the given differential equation to find the slope of the tangent line. Past exam questions are a helpful resource found in the Question Bank on AP Classroom for helping to build the connection between a contextual rate and the slope of a line. In particular, part (a) of AB4/BC4 on the 2017 exams is very similar to this question. Students had trouble with this then, as well, illustrating the need for continued review of the connections between past learning (slope of tangent line) and new learning (rate of change in temperature).
- In part (c) students needed to write an expression for $\frac{d^2M}{dt^2}$ and to use that information to reason that the tangent line approximation found in part (b) is an overestimate. Note that this is very similar to part (b) of AB4/BC4 on the 2017 exams.
- Finally, part (d) required students to solve a separable differential equation (Topics 7.6 and 7.7). Students typically perform well on this task, but in this case, they needed to be careful in handling the signs on the left-hand side of the equation. In addition to resources in the Course and Exam Description and AP Classroom for Topics 7.6 and 7.7, mastery of integration using substitution (Topic 6.9) was necessary to earn all points.

Question AB4/BC4

Topic: Graphical Analysis with L Hospital
Max Score: 9
Mean Score: AB4 2.70
Mean Score: BC4 4.03

What were the responses to this question expected to demonstrate?

In this problem the graph of f', which consists of a semicircle and two line segments on the interval $-2 \le x \le 8$, is provided. It is given that f is defined on the closed interval $[-2, 8]$, and that $f(2) = 1$.

In part (a) students were asked to reason whether f has a relative minimum, relative maximum, or neither at $x = 6$. A correct response will use the given graph to reason that f' does not change signs at $x = 6$, although $f'(6) = 0$. Therefore f has neither a relative maximum nor a relative minimum at this point.

In part (b) students were asked to find all open intervals where f is concave down. A correct response will reason that a function is concave down when its first derivative is decreasing, and therefore f is concave down on the intervals $(-2, 0)$ and $(4, 6)$.

In part (c) students were asked to find $\lim_{x \to 2} \frac{6f(x) - 3x}{x^2 - 5x + 6}$ or to show that this limit does not exist (and to justify their answer). A correct response will report that as x approaches 2, both the numerator and denominator of this ratio approach 0, and so L'Hospital's Rule applies. Using L'Hospital's Rule, the limit is shown to be 3.

In part (d) students were asked to find the absolute minimum value of f on the closed interval $[-2, 8]$ with a justification for their answer. A correct response will indicate the possible candidates for the location of the absolute minimum are the interval endpoints and the critical points $x = -1$, $x = 2$, and $x = 6$. A response could then reference work from part (a) to eliminate $x = 6$ as the location of a relative (or absolute) minimum, and could use the fact that the graph of f' changes from positive to negative at $x = -1$ in order to argue that a relative maximum occurs at $x = -1$. In addition, the given graph of f' indicates that $f'(x) \ge 0$ for $x > 2$, so the endpoint $x = 8$ cannot be the location of the absolute minimum value. The value of $f(2)$ is given in the stem of the problem, and using geometry, $f(-2) = 1 + \int_2^{-2} f'(x)\,dx = 3$. Therefore, the absolute minimum value of f on this closed interval is $f(2) = 1$. (Alternatively, a response could evaluate the function f at each of these five points and conclude that the absolute minimum is $f(2) = 1$.)

How well did the responses address the course content related to this question? How well did the responses integrate the skills required on this question?

Part (a)

- Overall responses to this part showed an understanding of the concept of, and requirements for, determining the location of relative extrema.
- The most common response to this prompt was the correct conclusion that there was neither a local maximum nor local minimum at $x = 6$ because $f'(x)$ does not change sign at this point.

Part (b)

- In most cases responses indicated an overall understanding of the concept of concavity and the reasoning for determining whether a function is concave up or concave down on a given interval.
- Most responses correctly listed the intervals $(-2, 0)$ and $(4, 6)$ as the intervals where the graph of f is concave down.
- The most common correct reason given for these intervals was that f' is decreasing on these intervals.
- Another common correct reason given was that the slope of f' is negative on these intervals.
- Some responses failed to connect the reason provided ($f''(x) < 0$) to the given graph of f'.

Part (c)

- A majority of the responses demonstrated that students have an understanding of how to apply L'Hospital's Rule and of the conditions necessary for using the rule.
- Most responses recognized the need for L'Hospital's Rule because they successfully evaluated $\lim\limits_{x\to 2} 6f(x) - 3x$ and $\lim\limits_{x\to 2} x^2 - 5x + 6,$ although several of these responses demonstrated "arithmetic with infinity" by writing $\lim\limits_{x\to 2} \dfrac{6f(x) - 3x}{x^2 - 5x + 6} = \dfrac{0}{0}.$ Correct responses either wrote the limits of the numerator and denominator separately or wrote that the limit was of the indeterminate form $\dfrac{0}{0}.$
- Most responses presented a limit of a ratio of derivatives, thereby applying L'Hospital's Rule. Some of these responses found an incorrect derivative of either $6f(x) - 3x$ or $x^2 - 5x + 6$ and, therefore, failed to find the correct limit.
- Most responses that found the limit of the ratio of correct derivatives continued on to find the correct value of the limit.

Part (d)

- Responses indicated a struggle to correctly apply all of the steps necessary to find an absolute extremum on a closed interval.
- Many responses did begin with an indication of the need to consider $f'(x) = 0,$ but quite a few never made this requirement clear.
- Most responses did consider the endpoints of the interval, but many had trouble correctly evaluating the function f at these endpoints (and at some of the critical points).
- Many responses considered some, but not all of the three critical points that lay in the interval. Responses may have felt that they did not need to mention $x = 6$ because this point was determined to be neither a local maximum nor a local minimum in part (a), but they should have referenced their work on part (a) in order to justify the answer.

What common student misconceptions or gaps in knowledge were seen in the responses to this question?

Common Misconceptions/Knowledge Gaps	*Responses that Demonstrate Understanding*
• In part (a) many responses used a global argument, claiming that "$f'(x) > 0$ to the left of 6 and $f'(x) > 0$ to the right of 6." This statement implies that $f'(x)$ is *always* non-negative, which is not true. • Some responses used a precalculus argument that "f doesn't change from increasing to decreasing or vice versa," without reference to the behavior of f'. • Many responses used vague language such as "the function doesn't change signs" or "it doesn't change signs."	• $f'(x)$ does not change sign at $x = 6$. • Because $f'(x)$ does not change sign at $x = 6$, the function f has neither a local maximum nor a local minimum at this point.
• In part (b) many responses reasoned that f is concave down on the presented intervals because $f''(x) < 0$ there, without tying the argument to the given graph of f'. • Some responses reported the interval $(-1, 2)$, where the graph of f' is negative. • Some responses made an argument based on the concavity of the given graph of f' and, therefore, concluded there were no intervals where the graph of f is concave down.	• The graph of f is concave down on $(-2, 0)$ and $(4, 6)$, because f'' = slope of f' is negative on these intervals. • The graph of f is concave down on $(-2, 0)$ and $(4, 6)$, because f' is decreasing on these intervals.
• In part (c) the most common mistake was to use L'Hospital's Rule without first checking that the limits of the numerator and of the denominator were both 0. • Some responses provided poor communication, writing $\lim_{x \to 2} \frac{6f(x) - 3x}{x^2 - 5x + 6} = \frac{0}{0}$. • Some responses failed to indicate the use of L'Hospital's Rule because they did not attach a limit to the presented ratio of derivatives. • The most common incorrect answer occurred in responses that used $f'(2) = 1$, thereby declaring $\lim_{x \to 2} \frac{6f'(x) - 3}{2x - 5} = \frac{3}{-1} = -3$.	• $\lim_{x \to 2} 6f(x) - 3x = 0$ and $\lim_{x \to 2} x^2 - 5x + 6 = 0$, therefore L'Hospital's Rule can be applied. • $\lim_{x \to 2} \frac{6f(x) - 3x}{x^2 - 5x + 6}$ is of indeterminate form $\frac{0}{0}$, therefore L'Hospital's Rule can be applied. • $\lim_{x \to 2} \frac{6f(x) - 3x}{x^2 - 5x + 6} = \lim_{x \to 2} \frac{6f'(x) - 3}{2x - 5}$ • $\lim_{x \to 2} \frac{6f'(x) - 3}{2x - 5} = \frac{6 \cdot 0 - 3}{2 \cdot 2 - 5} = \frac{-3}{-1} = 3$
• In part (d) many responses gave no indication of considering $f'(x) = 0$, and this could not be inferred because only some of the critical points were listed for consideration.	• An absolute maximum occurs at a critical point where $f'(x) = 0$ or at the endpoints of the interval.

• Very rarely did responses consider all three critical points in the interval $[-2, 8]$, although there was no consistency in which values were not considered. • Many responses did not use the initial condition $f(2) = 1$ when attempting to evaluate $f(x)$. • Some responses reported only a local minimum and did not consider the endpoints of the interval.	• $f'(x) = 0 \Rightarrow x = -1,\ x = 2,$ or $x = 6$ In part (a) it was determined there cannot be a maximum or minimum at $x = 6$. Because $f'(x)$ changes from positive to negative at $x = -1$, there cannot be a minimum there. We are given that $f(2) = 1$. • $f(-2) = 1 + \int_2^{-2} f'(x)\,dx = 3,$ $f(8) = 1 + \int_2^{8} f'(x)\,dx = 1 + 2 + (8 - 2\pi) = 11 - 2\pi$ Therefore, the absolute minimum of f in the interval $[-2, 8]$ is $f(2) = 1$.

Based on your experience at the AP® Reading with student responses, what advice would you offer teachers to help them improve the student performance on the exam?

- Teachers could focus on the reasoning and justification components of graphical analysis questions, which appear to be the most common difficulty for students with such questions.
- Teachers could have students practice writing descriptions of graphs of derivatives and the corresponding function graphs and second derivative graphs, always making clear distinctions between references to f, f', and f'', as well as avoiding the use of “it” and “the function” or “the graph.”

What resources would you recommend to teachers to better prepare their students for the content and skill(s) required on this question?

AB4/BC4 is a graphical analysis question and requires special attention to skill 4.A (Use precise mathematical language).

- Lessons and Student Handouts available in many topics on AP Classroom can help students to practice writing precisely. In this case, parts (a) and (b) both require care in communicating about the increasing/decreasing behavior of a function or its derivative over an interval rather than at a point. In part (a) it would be imprecise to say that $f'(x) > 0$ “before $x = 6$.” In part (b) students need to be careful to discuss the decreasing behavior of f' on two intervals. The Student Handout and Lesson Plan for Topic 5.4, which is linked on Topic 5.4 in AP Classroom, would be helpful in developing skill 4.A, along with understanding of relevant content.
- Scoring guidelines and Chief Reader Reports for past questions are essential resources. In this question, part (c) requires confirming that the condition for L’Hospital’s Rule has been satisfied (skill 3.C); that is, that the expression whose limit is being evaluated is indeterminate. The most common reason for a response attempting to do this but not earning the point is essentially underdeveloped use of appropriate mathematical symbols and notation (skill 4.C). In particular, the limit in this part, which is defined, should not be set equal to $\frac{0}{0}$, which is undefined. One correct symbolic representation of this confirmation is to separately evaluate the limits of the numerator and denominator of the given expression, as shown in the model solution in the scoring guidelines, found on the AP Calculus AB or BC exam page on AP Central. Past examples of similar questions are useful resources for modeling appropriate use of mathematical symbols and student practice (e.g., 2021 AB4/BC4 part (c) Scoring Guidelines and Chief Reader Report).
- Another useful resource for developing skill 4.C is the instructional strategy “Error Analysis,” described on page 206 of the Course and Exam Description.
- AP Daily videos for Topic 5.5 are a useful resource to help students to write complete justifications. High-quality feedback from teachers on formative assessment items is the ultimate resource for students. The scoring guidelines provided with topic questions and progress checks are a helpful resource for teachers.

Question AB5

Topic: Analysis of Functions
Max Score: 9
Mean Score: 4.46

What were the responses to this question expected to demonstrate?

In this problem students were given a table of selected values of the twice-differentiable functions f and g and of their first derivatives.

In part (a) students are asked to find $h'(7)$ for the function $h(x) = f(g(x))$. A correct response will use the chain rule to find $h'(x) = f'(g(x)) \cdot g'(x)$, then pull the appropriate values from the given table to find $h'(7) = 12$.

In part (b) students were told that k is a differentiable function such that $k'(x) = (f(x))^2 \cdot g(x)$ and were asked whether k is concave up or concave down at the point where $x = 4$. A correct response will use the product and chain rules to find $k''(x)$ and then evaluate $k''(4) = -40$ in order to determine that k is concave down at this point.

In part (c) the function $m(x) = 5x^3 + \int_0^x f'(t)\,dt$ is defined and students were asked to find $m(2)$. A correct response will use the Fundamental Theorem of Calculus to find $\int_0^2 f'(t)\,dt = f(2) - f(0)$, then use the given table to find $f(2)$ and $f(0)$. Finally, a correct response will combine the difference of these values with $5 \cdot 2^3$ to obtain $m(2) = 37$.

In part (d) students were asked whether this function m is increasing, decreasing, or neither at $x = 2$ and to provide a justification for their answer. A correct response will use the Fundamental Theorem of Calculus to find $m'(2) = 15 \cdot 2^2 + f'(2) = 52$ and realize that, because $m'(2)$ is positive, the function must be increasing in a neighborhood around $x = 2$.

How well did the responses address the course content related to this question? How well did the responses integrate the skills required on this question?

Part (a)

- Most responses demonstrated an understanding of the chain rule and were successful in pulling values from the given table to evaluate $h'(7) = f'(g(7)) \cdot g'(7)$.

Part (b)

- Some responses were able to correctly apply both the product and chain rules to find $k''(x)$.
- Such responses were generally able to evaluate the second derivative at $x = 4$ and conclude the function was concave down with a correct reason based on the sign of $k''(4)$.
- Many responses demonstrated knowledge that concavity was determined by the sign of k'', but some mislabeled the second derivative as k' and then determined the concavity based on the declared sign of $k'(4)$.
- Some responses used ambiguous language in the reasoning given and, therefore, did not provide sufficient justification.

Part (c)

- Roughly half of the responses were able to correctly evaluate the definite integral $\int_0^2 f'(t)\,dt$, but many did not demonstrate an understanding that $\int_0^x f'(t)\,dt = f(x) - f(0)$.
- As in parts (a) and (b), many responses presented communication errors by equating a variable expression to a constant value.

Part (d)

- Most responses successfully justified that m was increasing (or decreasing) at $x = 2$ based on the presented evaluation of $m'(2)$, but many did not demonstrate an understanding that $\frac{d}{dx}\int_0^x f'(t)\,dt = f'(x)$.

What common student misconceptions or gaps in knowledge were seen in the responses to this question?

Common Misconceptions/Knowledge Gaps	*Responses that Demonstrate Understanding*
• In part (a) the most common errors were in communication, equating two unequal expressions, e.g., $h'(x) = f'(g(7)) \cdot g'(7)$ or $h'(7) = f'(g(x)) \cdot g'(x)$. • There were some errors in applying the chain rule, e.g., $h'(x) = f(g(x)) \cdot g'(x)$ or $h'(x) = f'(g(x))$. • A few responses tried to use the product rule rather than the chain rule, e.g., $h'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$.	• $h'(x) = f'(g(x)) \cdot g'(x)$, $h'(7) = f'(g(7)) \cdot g'(7) = 12$
• In part (b) many responses had difficulty finding the second derivative of $k'(x)$. Some attempted to use the chain rule without the product rule, e.g., $k''(x) = 2f(x)f'(x)g'(x)$. Others tried to use the product rule without the chain rule, e.g., $k''(x) = 2f(x)g(x) + (f(x))^2 g'(x)$. • Many responses mislabeled $k''(x)$ and/or $k''(4)$, e.g., $k' = 2f \cdot f' \cdot g + (f)^2 \cdot g'$ or $k'(4) = 2 \cdot 4 \cdot 3 \cdot (-3) + 4^2 \cdot 2$. • There were a number of communication errors, e.g., $k''(x) = -40$, $k''(x) = 2 \cdot 4 \cdot 3 \cdot (-3) + 4^2 \cdot 2$, or $k''(4) = 2f(x) \cdot f'(x) \cdot g(x) + (f(x))^2 \cdot g'(x)$. • Many responses used ambiguous language such as "k is concave up at $x = 4$ because it (or the value) is positive."	• $k''(x) = 2f(x) \cdot f'(x) \cdot g(x) + (f(x))^2 \cdot g'(x)$ • $k''(4) = 2f(4) \cdot f'(4) \cdot g(4) + (f(4))^2 \cdot g'(4)$ • $k''(x) = 2f(x) \cdot f'(x) \cdot g(x) + (f(x))^2 \cdot g'(x)$, so $k''(4) = 2 \cdot 4 \cdot 3 \cdot (-3) + 4^2 \cdot 2 = -40$. • k is concave down at $x = 4$, because $k''(4) < 0$.

• In part (c) the most common misconception was in applying the Fundamental Theorem of Calculus, e.g., $\int_0^x f'(t)\,dt = f'(x)$ or $\int_0^x f'(t)\,dt = f(x)$. • Miscommunication errors were common, e.g., $m(x) = 37$ and $m(x) = 40 + \int_0^2 f'(t)\,dt$.	• $\int_0^x f'(t)\,dt = f(x) - f(0)$ • $m(x) = 5x^3 + \int_0^x f'(t)\,dt$, $m(2) = 5 \cdot 2^3 + \int_0^2 f'(t)\,dt$ $= 40 + f(2) - f(0) = 40 + 7 - 10 = 37$
• In part (d) there were many errors in applying the Fundamental Theorem of Calculus, e.g., $m'(x) = 15x^2 + f(x)$, $m'(x) = 15x^2 + f(x) - f(0)$, or $m'(x) = 15x^2 + f'(x) - f'(0)$. • Many responses used ambiguous language in the justifications, such as "m is increasing because it (or the slope) is positive." • Again, there were many communication errors, e.g., $m'(x) = 52$ or $m'(2) = 15x^2 + f'(2)$.	• $m'(x) = 15x^2 + f'(x)$ • m is increasing at $x = 2$, because $m'(2) > 0$. • $m'(x) = 15x^2 + f'(x)$, so $m'(2) = 15 \cdot 2^2 + f'(2) = 60 - 8 = 52$.

Based on your experience at the AP® Reading with student responses, what advice would you offer teachers to help them improve the student performance on the exam?

- Teachers could model precise and accurate communication, never displaying linkage errors or equating a variable expression to a constant value. Teachers could also require such accurate communication in all student work.
- Teachers could encourage students to carefully label their work so that they are unlikely to use terms such as "it" or "the function" in their reasoning or explanations. For example, a correct label of $m'(2) = 52$ makes it easier to argue that m is increasing at $x = 2$ because $m'(2) > 0$, rather than "because it is positive."
- Teachers could provide students with more practice finding derivatives that require use of both the product and chain rules or the quotient and chain rules. It would also be beneficial to incorporate multiple named functions such as $k(x) = (h(x))^2 \cdot g(x)$ or $p(x) = \frac{(f(x))^3}{r(x)}$.
- Teachers could provide students with opportunities to compare and contrast functions that may at first look similar. For example, compare the following: $\int_0^x f'(t)\,dt$, $\int_0^x f(t)\,dt$, $\frac{d}{dx}\int_0^x f'(t)\,dt$, and $\frac{d}{dx}\int_0^x f(t)\,dt$.

What resources would you recommend to teachers to better prepare their students for the content and skill(s) required on this question?

- This question requires identification (skill 1.C or 1.D) and application (skill 1.E) of appropriate mathematical rules and procedures. Most parts also involve identifying mathematical information from the given table, a numerical representation of functions (skill 2.B). To earn all points, responses must clearly present the use of the relevant rule(s). That is, points are not typically earned without supporting work. Relevant topics include 2.8, 3.1, and 6.7. AP Daily videos for each of these Topics provide guidance on how to show supporting work.
- It is typically necessary to show enough work that the form of the relevant derivative or integral is evident. The scoring notes for part (b) to this question illustrate several expressions containing a single error that would nonetheless earn the first point for showing the form of either the product rule or the chain rule. Model solutions, scoring notes, and Chief Reader Reports for past exam questions are excellent resources to help students to develop understanding and mastery.

- The need for special attention to application of the Fundamental Theorem of Calculus is noted. An older and especially rich resource mentioned in Topic 6.7 of the Course and Exam Description is *The Fundamental Theorem of Calculus*, a special focus professional development document. The teaching materials for students and discussion for teachers are still relevant today and extremely useful.

Question AB6

Topic: Implicit Differentiation with Related Rates
Max Score: 9
Mean Score: 2.77

What were the responses to this question expected to demonstrate?

This problem asked students to consider the curve defined by the equation $6xy = 2 + y^3$.

In part (a) students were asked show that $\frac{dy}{dx} = \frac{2y}{y^2 - 2x}$. A correct response will implicitly differentiate the equation $6xy = 2 + y^3$ with respect to x, then solve the resulting equation for $\frac{dy}{dx}$.

In part (b) students were asked to find the coordinates of a point on the curve at which the tangent line is *horizontal*, or to explain why no such point exists. A correct response will note that a horizontal tangent line must have $\frac{dy}{dx} = 0$, which requires $2y = 0$ and, therefore, $y = 0$. But if $y = 0$, using the given equation $6xy = 2 + y^3$ yields $6x \cdot 0 = 2$, which has no solution. Therefore, there is no point on this curve at which the tangent line is horizontal.

In part (c) students were asked to find the coordinates of a point on the curve at which the tangent line is *vertical*, or to explain why no such point exists. A correct response will begin by noting that such a point requires $y^2 - 2x = 0 \Rightarrow x = \frac{y^2}{2}$. Substituting into the equation $6xy = 2 + y^3$ yields $y = 1$ and then $x = \frac{1}{2}$, resulting in a vertical tangent line at the point $\left(\frac{1}{2}, 1\right)$.

In part (d) students were asked to find the value of $\frac{dy}{dt}$ at the instant when the particle is at the point $\left(\frac{1}{2}, -2\right)$, given that at that instant the particle's horizontal position is increasing at a rate of $\frac{dx}{dt} = \frac{2}{3}$. A correct response will implicitly differentiate the equation $6xy = 2 + y^3$ with respect to t and then solve the resulting equation for $\frac{dy}{dt}$ using $x = \frac{1}{2}$, $y = -2$, and $\frac{dx}{dt} = \frac{2}{3}$.

How well did the responses address the course content related to this question? How well did the responses integrate the skills required on this question?

Part (a)

- Most students performed well on this part of the problem. They correctly used the product rule to find the derivative of $6xy$ and the chain rule to find the derivative of y^3 and of y in the term $6xy$.
- Most responses were able to algebraically simplify their implicit differentiation to the form given in the statement of the problem.

Part (b)

- Overall responses demonstrated the correct conceptual connection between a tangent line being horizontal and a derivative $\frac{dy}{dx} = 0$.
- Most responses also recognized that substituting $y = 0$ into the equation $6xy = 2 + y^3$ resulted in the contradiction that $0 = 2$, although many students had trouble expressing this idea correctly. Most of the mistakes in this problem arose in the explanations and reasons given why no such point exists.

Part (c)

- Most responses that attempted to answer this part of the question were successful in determining that a vertical tangent line would mean $y^2 - 2x = 0$ (the denominator of $\frac{dy}{dx}$ would need to be 0).
- Some responses also recognized the need to then substitute either $x = \frac{y^2}{2}$ or $y = \sqrt{2x}$ into the given equation $6xy = 2 + y^3$, but many students had difficulty completing the algebra necessary to find both the x- and y-coordinates of the resulting point.
- In both parts (b) and (c) quite a few responses failed to realize the need to connect both the derivative $\frac{dy}{dx}$ and the curve $6xy = 2 + y^3$.

Part (d)

- Responses that recognized this as a standard related rates problem did very well on this part.
- Some responses recognized that this problem could also be viewed as a statement of the chain rule (using the derivative given in part (a)) and, therefore, easily found the correct value of $\frac{dy}{dt}$.
- Some responses struggled with distinguishing between $\frac{dx}{dt}$, $\frac{dy}{dt}$, and $\frac{dy}{dx}$. These responses had difficulty understanding the concept of differentiating with respect to a variable other than x.

What common student misconceptions or gaps in knowledge were seen in the responses to this question?

Common Misconceptions/Knowledge Gaps	*Responses that Demonstrate Understanding*
• In part (a) many responses had difficulty with the notation. For example, $\frac{dy}{dx}\left(6xy = 2 + y^3\right)$ or $\frac{dy}{dx}(6xy) = \left(2 + y^3\right)\frac{dy}{dx}$.	• $\frac{d}{dx}(6xy) = \frac{d}{dx}\left(2 + y^3\right)$
• In part (b) many responses used vague language. For example, some responses said "plugging y into the equation makes it undefined" or "the numerator must be 0." • Some responses made incorrect statements such as "when $y = 0$ the curve has no horizontal asymptote."	• Substituting $y = 0$ into the equation $6xy = 2 + y^3$ yields no solutions. • There is no point at which the line tangent to the curve is horizontal because if $y = 0$, then $6x(0) = 2 + 0^3$, which implies $0 = 2$.
• In part (c) there were also many imprecise responses, such as "the denominator must be 0." • Some responses did not make the connection that a point at which $\frac{dy}{dx}$ has a particular value or characteristic must necessarily be a point on the curve from which $\frac{dy}{dx}$ is obtained.	• The denominator of $\frac{dy}{dx} = \frac{2y}{y^2 - 2x}$ must be 0, or $y^2 - 2x = 0$.
• In part (d) several responses rewrote the equation of the curve as $6x = \frac{2 + y^3}{y}$, then failed to use the quotient rule when attempting to take the derivative of both sides. For example, $6\frac{dx}{dt} = \frac{3y^2\frac{dy}{dt}}{\frac{dy}{dt}} = 3y^2$. • Many responses failed to differentiate with respect to t (as opposed to x). For example, many responses wrote $6xy' + 6yx' = 3y^2y'$.	• $6\frac{dx}{dt} = \frac{y \cdot \left(3y^2\frac{dy}{dt}\right) - \left(2 + y^3\right)\frac{dy}{dt}}{y^2} = 2\left(\frac{y^3 - 1}{y^2}\right)\frac{dy}{dt}$ • $6x\frac{dy}{dt} + 6y\frac{dx}{dt} = 3y^2\frac{dy}{dt}$

Based on your experience at the AP® Reading with student responses, what advice would you offer teachers to help them improve the student performance on the exam?

- Teachers could emphasize the importance of proper notation and the value of good communication. In particular, when students are given a result and asked to show that it is true, the students must make sure that the work shown is accurate and supports the claim. If students observe the need to make an adjustment because the work is off from the desired answer, then the students should backtrack through the problem and find where the error originated. All of the steps in the problem must be consistent in leading to the given answer.

- Teachers could ensure that students have opportunities to work with free-response questions that require justifications and explanations of their reasoning.
- Teachers could incorporate the use of the notation $\frac{dy}{dx}$ or $\frac{dy}{dt}$ rather than y', particularly in situations involving related rates or implicit differentiation.
- Teachers could provide practice with problems involving implicit differentiation that go beyond merely finding $\frac{dy}{dx}$. It would be helpful to provide graphs of implicitly defined functions so that students could visualize situations like the one in this problem, where there are values of x and y such that $\frac{dy}{dx} = 0$, but (x, y) does not lie on the curve.

What resources would you recommend to teachers to better prepare their students for the content and skill(s) required on this question?

- Practicing implicit differentiation (Topic 3.2) offers opportunities for review and reinforcement of earlier topics. To differentiate the term $6xy$, for example, one must apply the product rule (Topic 2.8), with the implicit differentiation of y as $\frac{dy}{dx}$ a step along the way. If formative assessment (Topic Questions) suggests that particular students have gaps on understanding of earlier topics, AP Daily videos are a helpful resource for review. To master implicit differentiation requires skillful selection of differentiation procedures (Topic 3.5). *Categorizing Functions for Derivative Rules* is a resource provided with Topic 3.5 on AP Classroom.
- Topic 5.12 (Exploring Behaviors of Implicit Relations) offers an opportunity to help students pull together what they have learned in AP Calculus, as well as key prerequisite knowledge and skills. Because free-response questions on the AP Calculus exams are designed to assess content from across the course framework, these questions are often essential resources for uncovering opportunities for “just-in-time” review. For example, Free-Response Question AB5 on the 2021 AP Calculus exam would be a good jumping off point for uncovering the need for:
    - Review of differentiation techniques in implicit differentiation (part (a));
    - Review of prerequisite knowledge about horizontal (and, by extension, vertical) tangent lines in part (c);
    - Review of justification of relative extrema in part (d).
- For most effective use of these resources in class instruction, consider how the specific content of the question on a particular exam might be expanded beyond the one question. For example, focusing only on horizontal tangent lines in part (b) would represent a lost opportunity to review vertical tangent lines as well.
- Finally, this question highlights the importance of using precise mathematical language (skill 4.A). Key questions, sample activities, and sample instructional strategies to develop this skill are provided on page 219 of the Course and Exam Description.

Question BC2

Topic: Parametric Particle Motion – Acceleration-Speed-Distance
Max Score: 9
Mean Score: 5.20

What were the responses to this question expected to demonstrate?

In this problem students were told that a particle is moving along a curve so that its position at time t is $(x(t), y(t))$, with $y(t) = 2\sin t$, $\frac{dx}{dt} = e^{\cos t}$, and $0 \le t \le \pi$. Students were also told that at time $t = 0$, the particle is at position $(1, 0)$.

In part (a) students were asked to find the acceleration vector of the particle at time $t = 1$. This requires using a calculator to find the values $\left.\frac{d^2x}{dt^2}\right|_{t=1} = -1.444$ and $\left.\frac{d^2y}{dt^2}\right|_{t=1} = -1.683$.

In part (b) students were asked to find the first time t at which the speed of the particle is 1.5. A correct response will show the setup $\sqrt{\left(e^{\cos t}\right)^2 + (2\cos t)^2} = 1.5$ and then use a calculator to find the first time t in $[0, \pi]$ that satisfies this equation ($t = 1.254$).

In part (c) students were asked to find the slope of the line tangent to the particle's path at time $t = 1$ and then to find the position of the particle at this time. A correct response will indicate that the slope of the line tangent to the particle's path is $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$, then will use a calculator to find $\left.\frac{dy}{dx}\right|_{t=1} = 0.630$. The response will continue by noting that the x-coordinate of the position of the particle at time $t = 1$ is $x(0) + \int_0^1 \frac{dx}{dt}\,dt$ and will use a calculator to find that this value is 3.342.

In part (d) students were asked to find the total distance traveled by the particle over the time interval $0 \le t \le \pi$. A correct response will show the calculator setup of the integral of the particle's speed over this time interval, then evaluate the integral to find a total distance of 6.035.

How well did the responses address the course content related to this question? How well did the responses integrate the skills required on this question?

Part (a)

- Almost all responses recognized the need to use the derivatives of the given functions to find the acceleration of the particle.
- Quite frequently responses correctly identified the acceleration vector as $\langle x''(1), y''(1)\rangle$.
- Often responses used a calculator to find $x''(1)$, calculated $y'(x)$ and $y''(x)$ by hand, and then evaluated $y''(1)$ using a calculator.

Part (b)

- Nearly all responses presented a correct equation that could be solved in order to find the requested time.
- Responses indicated that most students knew how to use their calculator to solve an equation.
- Almost all responses that found more than one of the times when the particle's speed was 1.5 correctly committed to the first time.

Part (c)

- Most responses presented a correct value (exact or numerical) for the slope of the line tangent to the path of the particle.
- Virtually all of the responses that attempted to find the requested slope knew to compute a derivative. Most often the setup shown was some form of $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$. Some of these responses made no attempt to evaluate at $t = 1$.
- In finding the x-coordinate of the particle's position, most responses included the initial condition and most presented a correct value. In some cases, the initial condition was subtracted from the value of $\int_0^1 \frac{dx}{dt}\,dt$ rather than added to it.

Part (d)

- Responses indicated that a majority of students knew that finding the total distance required integrating an expression for the particle's speed.
- Nearly all responses successfully used a calculator to evaluate the presented integral expression.

What common student misconceptions or gaps in knowledge were seen in the responses to this question?

Common Misconceptions/Knowledge Gaps	*Responses that Demonstrate Understanding*
• In part (a) some responses reported a velocity vector, $\langle x'(1), y'(1)\rangle$, rather than the acceleration vector. • Some responses reversed the x- and y-coordinates of the acceleration vector, e.g., $\langle y''(1), x''(1)\rangle$. • A few responses equated a variable expression with a constant, e.g., $\langle x''(t), y''(t)\rangle = \langle -1.444, -1.683\rangle$. • Some responses incorrectly differentiated either $e^{\cos t}$ or $2\sin t$ by hand, which resulted in subsequent differentiation errors and incorrect values.	• Velocity $= \langle x'(t), y'(t)\rangle = \left\langle e^{\cos t}, 2\cos t\right\rangle$ Acceleration $= \langle x''(t), y''(t)\rangle = \left\langle -\sin t \cdot e^{\cos t}, -2\sin t\right\rangle$ • Acceleration $= \langle x''(1), y''(1)\rangle = \langle -1.444, -1.683\rangle$ • $\langle x''(1), y''(1)\rangle = \langle -1.444, -1.683\rangle$ or $\left.\left\langle \frac{d^2x}{dt^2}, \frac{d^2y}{dt^2}\right\rangle\right\|_{t=1} = \left\langle -e^{\cos 1}\sin 1, -2\sin 1\right\rangle$ • $\frac{d}{dt}e^{\cos t} = -\sin t \cdot e^{\cos t}$, $\frac{d}{dt}2\sin t = 2\cos t$
• In part (b) some responses found the speed at time $t = 1.5$ rather than solving the speed $= 1.5$ for t. • Some responses used the TRACE function on a calculator and consequently found a solution that was not correct to three digits after the decimal. • Occasionally responses presented "linkage errors" in communication, e.g., $\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = 1.5 = 1.254.$	• $\sqrt{\left(e^{\cos t}\right)^2 + (2\cos t)^2} = 1.5 \Rightarrow t = 1.254$ • $t = 1.254472$ • $\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = 1.5 \Rightarrow t = 1.254$
• In part (c) several responses reported the tangent line equation without identifying the slope of the line.	• $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2\cos t}{e^{\cos t}}$

• Occasionally, a response evaluated $\frac{dy}{dx}$ at an incorrect value of t, such as $t = 1.5$. • Some responses presented only an indefinite integral in their setup for the particle's x-coordinate. • Many responses failed to include a differential (dt) in the integral setup for $x(1)$. Frequently this resulted in an ambiguous setup, $\int_0^1 e^{\cos t} + 1$, which could not be given credit. • Some responses presented "linkage errors" in communication, e.g., $x(1) = \int_0^1 e^{\cos t}\, dt = 1 + 2.342$.	• $\left.\frac{dy}{dx}\right\rvert_{t=1} = \frac{2\cos 1}{e^{\cos 1}} = 0.630$ • $x(1) = 1 + \int_0^1 e^{\cos t}\, dt = 3.342$ • $x(1) = \int_0^1 e^{\cos t}\, dt + 1 = 3.342$ • $\int_0^1 e^{\cos t}\, dt = 2.342,\ x(1) = 1 + 2.342 = 3.342$
• In part (d) several responses presented the correct setup but evaluated incorrectly, e.g., $\int_0^\pi \sqrt{\left(e^{\cos t}\right)^2 + (2\cos t)^2}\, dt = 13.447$. (This value is obtained by calculating $\int_0^\pi \left(e^{\cos t}\right)^2 + (2\cos t)^2\, dt$.) • Some responses presented the integral $\int_0^\pi \sqrt{1 + \left(\frac{2\cos t}{e^{\cos t}}\right)^2}\, dt = 6.783$.	• $\int_0^\pi \sqrt{\left(e^{\cos t}\right)^2 + (2\cos t)^2}\, dt = 6.035$ • Total distance traveled $= \int_0^\pi \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^\pi \sqrt{\left(e^{\cos t}\right)^2 + (2\cos t)^2}\, dt$ $= 6.035$

Based on your experience at the AP® Reading with student responses, what advice would you offer teachers to help them improve the student performance on the exam?

- Teachers could repeatedly remind students of the importance of understanding and properly using standard mathematical notation rather than creating new notation to communicate their ideas.
- Teachers could ask students to write specific integrands using defined functions and limits rather than relying on rote formula expressions.
- Teachers could model precise and accurate communication, never displaying linkage errors or equating a variable expression to a constant value.
- Teachers could have students practice using a calculator to solve an equation (without using the TRACE function).

What resources would you recommend to teachers to better prepare their students for the content and skill(s) required on this question?

- A favorite resource for teaching and learning about parametric particle motion is *Vectors: A Curriculum Module for AP® Calculus BC*. Through a series of six days of lessons, author Nancy Stephenson shares resources and practice problems that take students through required prerequisite knowledge about parametric equations, the calculus of parametric equations, a review of rectilinear motion (AP Calculus AB content from Units 4 and 8), and particle motion along a curve, including vectors (Unit 9). Having mastered the material in this resource, a student would be fully prepared to answer BC2: Find an acceleration vector (part (a)), work with speed (part (b)), find the slope of a line tangent to the path and the *x*-coordinate of the position of the particle (part (c)), and find the total distance traveled by the particle over an interval (part (d)).
- The resource *AP Calculus: Use of Graphing Calculators* can help to develop students' calculator skills.
    - Students should always have their calculators in radian mode when taking an AP Calculus exam or working in calculus class. Establishing this habit early and practicing all year can help students to earn all of the points merited by their mastery of calculus.
    - As mentioned in the classroom resource *Commentary on the Instructions for the Free-Response Section of the AP® Calculus Exams*, students should also be careful to show supporting work. In particular, if a calculator is used to determine a value, a response must show the setup entered into the calculator and use standard mathematical notation. For example, in part (a) students were prompted to "show the setup for your calculations" and were expected to do so to earn full credit.
    - Students should be careful to apply appropriate rounding procedures (skill 4.E). This includes rounding (or truncating) final answers to at least three places after the decimal and storing as many places as possible for intermediate values. On page 220 of the Course and Exam Description, teachers can find key questions and instructional strategies to help students to master this skill.

Question BC5

Topic: Area-Volume with Improper Integral and Integration by Parts
Max Score: 9
Mean Score: 4.71

What were the responses to this question expected to demonstrate?

In this problem students were given a figure showing a shaded region bounded by the graphs of functions f and g for $0 \le x \le 3$. Students were told that $g(x) = \frac{12}{3 + x}$ for $x \ge 0$ and that f is differentiable with $f(3) = 2$ and $\int_0^3 f(x)\, dx = 10$.

In part (a) Students were asked to find the area of the shaded region. This requires setting up and evaluating $\int_0^3 (f(x) - g(x))\, dx$. To evaluate, a student will need to separate into two integrals, $\int_0^3 f(x)\, dx - \int_0^3 g(x)\, dx$, and find an antiderivative for the function g. A correct response will provide an answer of $10 - 12(\ln|3 + x|)\Big|_0^3 = 10 - 12(\ln 6 - \ln 3)$.

In part (b) students were asked to evaluate the improper integral $\int_0^\infty (g(x))^2\, dx$ or to show that the integral diverges. A correct response will employ correct limit notation to rewrite the improper integral with a variable upper limit, find the correct antiderivative ($\int \frac{144}{(3 + x)^2}\, dx = -\frac{144}{(3 + x)}$), and continue the correct limit notation to find a value of 48.

In part (c) students were asked to find the value of $\int_0^3 h(x)\, dx$ given that $h(x) = x \cdot f'(x)$. A correct response will recognize the need to use integration by parts to find $\int_0^3 h(x)\, dx = x \cdot f(x)\Big|_0^3 - \int_0^3 f(x)\, dx = 6 - 0 - 10 = -4$.

How well did the responses address the course content related to this question? How well did the responses integrate the skills required on this question?

<u>Part (a)</u>

- Most responses showed an understanding of the connection between the definite integral and area between two curves. Some responses used an integrand that would represent a volume of revolution or a volume by cross section. Some responses used an incorrect integrand of $(10 - g(x))$, while others incorrectly reversed the integrand.
- Most responses correctly determined the antiderivative of $\frac{12}{3 + x}$. Many responses used u-substitution to correctly find this antiderivative; however, some responses did not change the limits of integration to take into account the new interval of integration in terms of u.
- A majority of responses that displayed the correct antiderivative evaluated the expression using the bounds correctly. Some responses attempted to simplify their answer using properties of logarithms with mixed results, and others had difficulty distributing a negative across parentheses, for example: $10 - 12\ln|x + 3|\Big|_0^3 = 10 - 12\ln 6 - 12\ln 3$.

Part (b)

- Most of the responses recognized that a limit is needed to evaluate an improper integral, but many responses were unable to correctly use limit notation throughout the response. Errors included an initial presentation of limit notation that was subsequently dropped, limit notation used too late at the end of the evaluation, a mismatch between the limiting variable(s) and the variable(s) used as an upper bound, and arithmetic performed with infinity (e.g., $\left[-\frac{144}{3+x}\right]_0^\infty$ or $-\frac{144}{3+\infty}+48$).
- Many responses correctly determined the antiderivative of $\frac{144}{(3+x)^2}$, often using u-substitution to correctly find this antiderivative; however, some responses did not change the limits of integration to take into account the new interval of integration in terms of u.
- Some responses did not antidifferentiate correctly, attempting to use the natural logarithm or a partial fraction decomposition.
- A majority of responses that displayed the correct antiderivative evaluated the expression correctly.

Part (c)

- Many responses recognized the need to use integration by parts and correctly applied this technique of integration. Some responses did not correctly antidifferentiate the literal function $f'(x)$, while others incorrectly used addition instead of subtraction when applying this technique.
- Some responses attempted to antidifferentiate by using a product of antiderivatives.
- Many responses showed a partial understanding of how to correctly handle the limits of integration on a definite integral evaluated using integration by parts. Because this understanding was only partial, some responses presented incorrect evaluations.

What common student misconceptions or gaps in knowledge were seen in the responses to this question?

Common Misconceptions/Knowledge Gaps	*Responses that Demonstrate Understanding*
• In part (a) many responses provided an integral for a volume rather than the area, e.g., $\int_0^3 (f(x)-g(x))^2\,dx$ or $\int_0^3 \left((f(x))^2-(g(x))^2\right)dx.$	• Area $=\int_0^3 (f(x)-g(x))\,dx$
• Some responses confused the integrand and the value of the integral, e.g., $\int_0^3 (10-g(x))\,dx.$	• Area $=\int_0^3 (f(x)-g(x))\,dx = 10-\int_0^3 g(x)\,dx$
• Several responses presented incorrect bounds when integrating using a u-substitution, e.g., $\int_0^3 \frac{12}{3+x}\,dx = \int_0^3 \frac{12}{u}\,du.$	• $\int_0^3 \frac{12}{3+x}\,dx = \int_3^6 \frac{12}{u}\,du$
	• $-12\ln 6 + 12\ln 3 = -12\ln 2 = 12\ln\frac{1}{2}$

• Many responses had trouble simplifying logarithms, e.g., $-12\ln 6 + 12\ln 3 = -12\ln 3$, $-12\ln 6 + 12\ln 3 = -12\ln \frac{1}{2}$, or $-12\ln 6 + 12\ln 3 = -12\ln 18$.	
• In part (b) many responses presented incorrect integration of terms with negative exponents, e.g., $\int_0^b (3+x)^{-2}\,dx = -\frac{1}{3}(3+x)^{-3}\Big\vert_0^b$ or $\int_0^b \frac{1}{(3+x)^2}\,dx = \frac{1}{(3+x)^3}\Big\vert_0^b$. • Other responses reported a logarithmic antiderivative, e.g., $\int_0^b \frac{1}{x^2+6x+9}\,dx = \ln\left\vert x^2+6x+9\right\vert\Big\vert_0^b$. • As in part (a), many responses presented incorrect bounds when using a u-substitution, e.g., $\int_0^b \frac{144}{(3+x)^2}\,dx = \int_0^b \frac{144}{u^2}\,du$.	• $\int_0^b (3+x)^{-2}\,dx = -(3+x)^{-1}\Big\vert_0^b$ • $\int_0^b \frac{144}{(3+x)^2}\,dx = \int_3^{b+3} \frac{144}{u^2}\,du$
• In part (c) many responses failed to correctly use integration by parts, e.g., $\int x \cdot f'(x)\,dx = x \cdot f(x) + \int f(x)\,dx$, $\int x \cdot f'(x)\,dx = x \cdot f(x) - \int f'(x)\,dx$, or $\int h(x)\,dx = x \cdot f'(x) + \int f'(x)\,dx$. • Many responses tried to integrate by multiplying two antiderivatives, e.g., $\int x \cdot f'(x)\,dx = \frac{1}{2}x^2 \cdot f(x)$.	• $u = x \quad dv = f'(x)\,dx$ $du = dx \quad v = f(x)$ $\int x \cdot f'(x)\,dx = x \cdot f(x) - \int f(x)\,dx$

Based on your experience at the AP® Reading with student responses, what advice would you offer teachers to help them improve the student performance on the exam?

- Teachers could look for ways to help students recognize situations when a particular technique of integration (substitution, by parts, partial fractions, etc.) is appropriate. Teachers should revisit this idea when using integration in a variety of contexts.
- Teachers could provide significant practice integrating by parts, expecting a clear delineation of both u and dv. Students should be required to provide unambiguous communication, including appropriate labels on intermediate steps, particularly if they are using tabular integration by parts.
- Teachers could provide more demonstrations of and practice with using bounds of integration for integrals that will be evaluated after using integration by parts. Students should be required to write bounds of evaluation on uv and bounds of integration on $\int v\,du$.
- Teachers could provide more demonstrations of and practice with using correct limit notation in evaluating improper integrals, making sure to carry the correct notation throughout the entire problem. Teachers should stress writing clear and logical mathematical work, paying particular attention to expressions involving limits and the proper placement of bounds of integration.

- Teachers should repeatedly provide opportunities for students to practice working with negative exponents in the context of antidifferentiation.

What resources would you recommend to teachers to better prepare their students for the content and skill(s) required on this question?

- This question begins with content included in the AP Calculus AB course framework: *Finding the Area Between Curves Expressed as Functions of* x (Topic 8.4). It is important that students have opportunities to refresh and strengthen their mastery of AB content, even as they build on that with BC topics.
- Scoring notes provided with the scoring guidelines for AP Calculus free-response questions are an excellent resource for teachers and students to develop understanding of sometimes subtle issues associated with certain questions.
- The scoring notes to part (a) of this question, for example, consider several ways students might approach the order of subtraction for the two functions in the integrand for the solution.
  - Because $f(x)$ is above $g(x)$ for the entire interval, an integrand of $f(x) - g(x)$, as presented in the model solution is most likely to be presented.
  - If this had been a calculator active question (or if determining which function was above the other were more difficult), more students might have written an integrand of $|f(x) - g(x)|$ or $|g(x) - f(x)|$.
  - Along the same lines, some students might have written $g(x) - f(x)$ and then recognized the reversal in order of subtraction when a negative sign emerged, correcting it with a simple sign change.

  The scoring notes handle each of these potentially correct cases, including consideration of how each might go wrong.
- The best use of the scoring notes as an instructional resource is not to focus on how to score specific errors. Rather, it is important to use the scoring notes to surface potential errors, so that they can be avoided. The scoring notes to part (b), for example, call out the importance of using limit notation throughout the problem and not to include "arithmetic with infinity." The model solution shows students how they should be presenting their work with an improper integral.

Question BC6

Topic: Taylor Polynomial with Lagrange Error Bound
Max Score: 9
Mean Score: 4.21

What were the responses to this question expected to demonstrate?

In this problem students were told that the function f has derivatives of all orders for all real numbers and that $f(0) = 2$, $f'(0) = 3$, $f''(x) = -f\left(x^2\right)$, and $f'''(x) = -2x \cdot f'\left(x^2\right)$.

In part (a) students were asked to find $f^{(4)}(x)$ and then to write the fourth-degree Taylor polynomial for f about $x = 0$. A correct response will use the product and chain rules to find $f^{(4)}(x) = -2 \cdot f'\left(x^2\right) + (-2x) f''\left(x^2\right) \cdot 2x$. The response will then evaluate the first four derivatives of f at $x = 0$ and use these values to write the fourth-degree Taylor polynomial $T_4(x) = f(0) + f'(0) \cdot x + \frac{f''(0)}{2!} \cdot x^2 + \frac{f'''(0)}{3!} \cdot x^3 + \frac{f^{(4)}(0)}{4!} \cdot x^4$, which is $T_4(x) = 2 + 3x - x^2 - \frac{1}{4}x^4$.

In part (b) students were asked to use the Lagrange error bound to show that the approximation of $f(0.1)$ found using the fourth-degree Taylor polynomial is within $\frac{1}{10^5}$ of the exact value, given that $\left|f^{(5)}(x)\right| \le 15$ for $0 \le x \le 0.5$. A correct response will indicate that the Lagrange error bound limits the absolute value of the difference between the approximation and the exact value to $\frac{\max\limits_{0 \le x \le 0.1}\left|f^{(5)}(x)\right|}{5!} \cdot (0.1)^5 \le \frac{15}{120} \cdot \frac{1}{10^5}$ which is less than $\frac{1}{10^5}$.

In part (c) students were told that g is a function with $g(0) = 4$ and $g'(x) = e^x f(x)$ and asked to write the second-degree Taylor polynomial for g about $x = 0$. A correct response will use the product rule to find $g''(0) = e^0 \cdot f'(0) + e^0 \cdot f(0) = 5$, evaluate $g'(0) = e^0 f(0) = 2$, and then put these two values together with the given value of $g(0) = 4$ to write the polynomial $T_2(x) = 4 + 2x + \frac{5}{2!}x^2$.

How well did the responses address the course content related to this question? How well did the responses integrate the skills required on this question?

Part (a)

- Most responses recognized the need to find the fourth derivative of a function in order to find the x^4-term in a fourth-degree Taylor polynomial.
- A significant number of responses did not correctly find the fourth derivative of f using the product rule.
- Most responses demonstrated an understanding of how to create the fourth-degree Taylor polynomial once the appropriate derivatives were known.

Part (b)

- Most responses showed a familiarity with the structure of the Lagrange error bound and most were able to simplify the bound to demonstrate that it was less than $\frac{1}{10^5}$.

Part (c)

- Most responses demonstrated an understanding of the need to find the second derivative of $g(x)$ in order to create the second-degree Taylor polynomial.
- As in part (a), there were some mistakes made in using the product rule to find $g''(x)$, but most responses presented a second-degree Taylor polynomial with a coefficient of $\frac{g''(0)}{2}$ for the x^2-term, using the presented expression for $g''(x)$.

What common student misconceptions or gaps in knowledge were seen in the responses to this question?

Common Misconceptions/Knowledge Gaps	*Responses that Demonstrate Understanding*
• In part (a) some responses failed to use the chain rule when attempting to find $f^{(4)}(x)$, e.g., $f^{(4)}(x) = -2x \cdot f''\left(x^2\right) - 2 \cdot f'\left(x^2\right)$. Other responses made errors in using the product rule. • Some responses used poor communication in equating a variable expression to a constant, e.g., $f^{(4)}(x) = f^{(4)}(0)$. • In parts (a) and (c) some responses presented an incorrect number of terms or a Taylor series rather than polynomials.	• $f^{(4)}(x) = -2x \cdot f''\left(x^2\right) \cdot 2x - 2 \cdot f'\left(x^2\right)$ • $f^{(4)}(x) = -2 \cdot f'\left(x^2\right) - 2x \cdot f''\left(x^2\right) \cdot 2x$ $f^{(4)}(0) = -2 \cdot f'(0) + 0 = -2 \cdot 3$ —OR— $f^{(4)}(x)\Big\vert_{x=0} = -2 \cdot f'(0) + 0 = -2 \cdot 3$ • $T_4(x) = 2 + 3x - x^2 - \frac{1}{4}x^4$
• In part (b) some responses failed to tie the Lagrange error bound to the absolute difference between the approximate and the exact value of $f(0.1)$. • Some responses provided the Lagrange error bound but did not indicate that the bound was less than $\frac{1}{10^5}$. • A few responses simplified the Lagrange error bound incorrectly, e.g., $\frac{15}{120}(0.1)^5 = \frac{3(0.1)^5}{40}$.	• $\lvert T_4(0.1) - f(0.1) \rvert \le \frac{\max\limits_{0 \le x \le 0.1} \left\lvert f^{(5)}(x) \right\rvert}{5!} \cdot (0.1)^5$ • $\frac{\max\limits_{0 \le x \le 0.1} \left\lvert f^{(5)}(x) \right\rvert}{5!} \cdot (0.1)^5 \le \frac{15}{120} \cdot \frac{1}{10^5} < \frac{1}{10^5}$
• In part (c) many responses presented errors in finding $g''(x)$, e.g., $g''(x) = e^x \cdot f'(x)$. • Another common knowledge gap was equating a variable expression to a constant, e.g., $g''(x) = g''(0)$.	• $g''(x) = e^x \cdot f(x) + e^x \cdot f'(x)$ • $g''(x) = e^x \cdot (f(x) + f'(x))$ $g''(0) = e^0 \cdot (f(0) + f'(0))$ —OR— $g''(x)\Big\vert_{x=0} = g''(0)$

Based on your experience at the AP® Reading with student responses, what advice would you offer teachers to help them improve the student performance on the exam?

- Students would benefit by practicing finding Taylor polynomials of various degrees for composite functions such as $f\left(x^2\right)$ and for products of functions such as $e^x \cdot f(x)$. More practice with such functions will reduce the likelihood of a mistake in finding higher-order derivatives.
- Teachers should have students be careful with notation at all times. Many responses mistakenly wrote $g''(x) = g''(0)$ and/or $f^{(4)}(x) = f^{(4)}(0)$, when correct notation is $g''(x)\big|_{x=0} = g''(0)$ or a similar statement to demonstrate that these two expressions are not always equal.
- Students also need additional support with finding error bounds and constructing valid comparisons with inequality symbols. (For example, "$\text{Error } = \left|T_4(0.1) - f(0.1)\right| \le \text{ Lagrange Error Bound}$" rather than "$\text{Error } = \left|T_4(0.1) - f(0.1)\right| < \text{ Lagrange Error Bound.}$") Teachers could provide students with repeated practice communicating the difference between "error" (in an approximation) and an "error bound" and make sure that students are precise and clear with their notation and statements.

What resources would you recommend to teachers to better prepare their students for the content and skill(s) required on this question?

- This question explores Taylor polynomial approximation for a function f (parts (a) and (b)) and finding a Taylor polynomial for a function g (part (c)). AP Daily videos for Topics 10.11 and 10.12 would be helpful resources for teachers and students.
- For an in-depth discussion of Taylor Series approximations, please see Ruth Dover's article on pages 77–79 of the College Board publication *Approximation*. Also in that resource, please find Jim Hartman's Instructional Unit: Taylor Polynomial Approximation of Functions (pages 80–91).