Question 6

Intent of Question

This question was designed to evaluate a student's ability to make inferences for simple linear regression models. Interpreting model parameters and comparing and contrasting different models are important skills that are also being assessed. Finally, a multiple regression model with a special variable, an indicator variable, is introduced to investigate whether the relationship between the predictor and response variable differs for two different groups of people. Students are asked to sketch the estimated line for both groups and interpret the estimated parameters in the multiple regression model.

Solution

Part (a):

The value 1.080 estimates the *average* increase (in feet) in the perceived distance for each additional foot in actual distance between the two objects.

Part (b):

The model with zero intercept makes more intuitive sense in this particular situation. If the two objects are placed side by side (so the actual distance is zero), then we would expect the subjects to say that the distance between the objects is zero.

Part (c):

Let β denote the true slope between the perceived distances and the actual distances. The researcher's hypothesis is equivalent to $\beta > 1$. Thus, we want to conduct a hypothesis test for the slope parameter.

Step 1: States a correct pair of hypotheses:

$H_0 : \beta = 1$
$H_a : \beta > 1$

Step 2: Correct mechanics, including the value of the test statistic and *p*-value (or rejection region).

This is a *t*-test of a slope.

$$t = \frac{b - \beta}{s_b} = \frac{1.102 - 1}{0.393} = 0.260$$

$$\text{df} = 40 - 1 = 39$$

$p\text{-value} = P(t > .260) = 0.398$

Question 6 (continued)

Step 3: States a correct conclusion in the context of the problem, using the result of the statistical test.

Since the *p*-value 0.398 is greater than 0.05, we cannot reject H_0. That is, we do not have statistically significant evidence to conclude that the subjects overestimate the distance with the magnitude of the overestimation increasing as the actual distance increases.

Part (d):

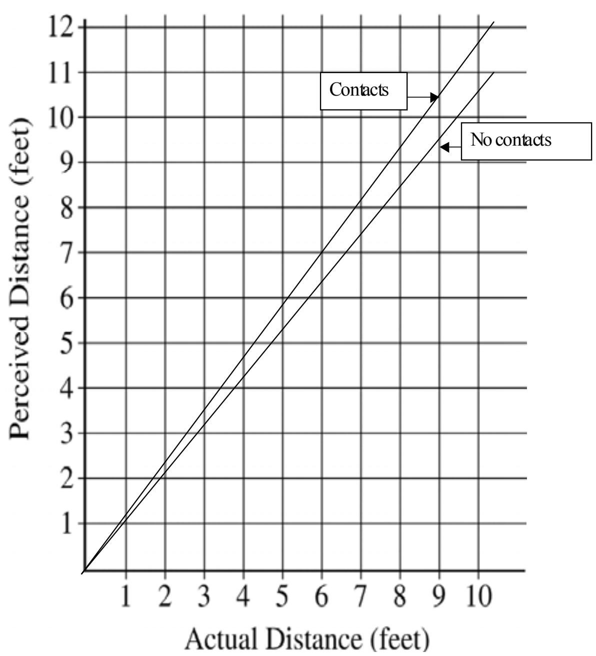
According to Model 3, the estimated models for the two groups are:

Contact wearers (*contact* = 1):

perceived distance = 1.05 (*actual distance*) + 0.12 (*actual distance*)
= 1.17 (*actual distance*)

Noncontact wearers (*contact* = 0):

perceived distance = 1.05 (*actual distance*)



Part (e):

Model 3 allows prediction of perceived distance separately for contact wearers and for noncontact wearers. The value of 1.05 estimates the average increase (in feet) in the perceived distance for each one-foot increase in actual distance for the population of noncontact wearers. The value of 0.12 estimates the *additional* increase (in feet) in the average perceived distance for each one-foot increase in actual distance for the contact wearers.

Question 6 (continued)

Scoring

Parts (a) and (b) are combined and scored as essentially correct (E), partially correct (P), or incorrect (I). Parts (c), (d), and (e) are scored as essentially correct (E), partially correct (P), or incorrect (I).

Parts (a) and (b) combined is scored as essentially correct (E) if both parts are correct.

Parts (a) and (b) combined is scored as partially correct (P) if:
one part is correct and the other part is incorrect;
OR
one part is correct and the other part is partially correct;
OR
both parts are partially correct.

Part (a) and (b) combined is scored as incorrect (I) if one part is partially correct.

Notes:

Part (a) is scored as partially correct if there is no word that makes it clear that 1.080 is not a deterministic increase.

Part (a) is scored as incorrect if the response:
- ignores the intercept and implies proportionality: for each foot of actual distance between the two objects, the subject perceives about 1.080 feet;
- consists of the equation rewritten in words.

Part (b)

Additional correct statement:
- The intercept is clearly not statistically significant, so the simpler model that includes only the slope is reasonable.

Partially correct statements:
- The SE for Model 2 is so large that Model 2 does not seem reasonable.
- The interpretation of the slope is straightforward if there is a 0 intercept: the percentage error is *slope* – 1 or 10.2 percent.
- The slope for Model 2 is farther above 1 than the slope for Model 1 and so more in line with the researcher's hypothesis.

Incorrect statements:
- Having one SE is better than having two.
- It is simpler/easier/shorter/more accurate to have just one coefficient.

Question 6 (continued)

Part (c) is scored as:

Essentially correct (E) if three steps are correct.

Partially correct (P) if two steps are correct.

Incorrect (I) if one step is correct.

Notes:

- Hypotheses: the hypotheses step is incorrect if the alternative hypothesis is two-sided, or if the null hypothesis is $\beta = 0$. (It is not necessary to define β.)
- Computation: if the computation includes division by $\sqrt{40}$, the computation step is incorrect.
- Conclusion: a conclusion with no context is incorrect.

Part (d) is scored as essentially correct (E) if both estimated regression lines are graphed correctly and at least one is labeled.

Part (d) is scored as partially correct (P) if:

- the lines are graphed correctly but neither is labeled;
  OR
- the graphs consist of unconnected dots.

Part (d) is scored as incorrect (I) if:

- the two lines on the grid have the same slope;
  OR
- one line is plotted correctly and one line is not.

Part (e) is scored as essentially correct (E) if the response includes a correct interpretation of the estimated coefficients, 1.05 and 0.12. Unlike in part (a) there is no *y*-intercept, so this statement is correct: "For each foot of actual distance between the two objects, a noncontact wearer perceives about 1.05 feet, and a contact wearer will perceive about an additional 0.12 feet."

Part (e) is scored as partially correct (P) if:

- the response includes a correct interpretation of just one of the two coefficients;
  OR
- the response includes a correct interpretation of 1.05 and 1.05 + 0.12 = 1.17 but doesn't include a separate interpretation of 0.12;
  OR
- no numbers are mentioned, but it is made clear that both groups overestimate the distance *AND* that contact wearers overestimate more than do noncontact wearers.

Part (e) is scored as incorrect (I) if:

- the response says only that 1.05 and 0.12 are "slopes of regression lines";
  OR
- only the SEs of the coefficients, 0.357 and 0.032, are interpreted.

Question 6 (continued)

Each essentially correct (E) response counts as 1 point; each partially correct (P) response counts as ½ point.

4 **Complete Response**

3 **Substantial Response**

2 **Developing Response**

1 **Minimal Response**

If a response is between two scores (for example, 2½ points), use a holistic approach to determine whether to score up or down depending on the strength of the response and communication.

STATISTICS
SECTION II
Part B
Question 6
Spend about 25 minutes on this part of the exam.
Percent of Section II grade—25

Directions: Show all your work. Indicate clearly the methods you use, because you will be graded on the correctness of your methods as well as on the accuracy and completeness of your results and explanations.

6. A study was designed to explore subjects' ability to judge the distance between two objects placed in a dimly lit room. The researcher suspected that the subjects would generally overestimate the distance between the objects in the room and that this overestimation would increase the farther apart the objects were.

   The two objects were placed at random locations in the room before a subject estimated the distance (in feet) between those two objects. After each subject estimated the distance, the locations of the objects were rerandomized before the next subject viewed the room.

   After data were collected for 40 subjects, two linear models were fit in an attempt to describe the relationship between the subjects' perceived distances (y) and the actual distance, in feet, between the two objects.

   $$\text{Model 1: } \hat{y} = 0.238 + 1.080 \times (\text{actual distance})$$

   The standard errors of the estimated coefficients for Model 1 are 0.260 and 0.118, respectively.

   $$\text{Model 2: } \hat{y} = 1.102 \times (\text{actual distance})$$

   The standard error of the estimated coefficient for Model 2 is 0.393.

   (a) Provide an interpretation in context for the estimated slope in Model 1.

For every one foot further away the two objects actually are, our best estimate is that the perceived distance will increase by 1.080 feet on average.

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(b) Explain why the researcher might prefer Model 2 to Model 1 in this context.

The researcher may believe that the true relationship is directly linear, and that if the objects were in the same place they would not be perceived as 0.238 feet apart or anything near that large.

(c) Using Model 2, test the researcher's hypothesis that in dim light participants overestimate the distance, with the overestimate increasing as the actual distance increases. (Assume appropriate conditions for inference are met.)

$H_0: \beta = 1$

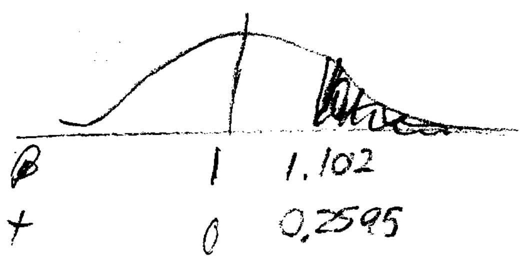
$H_A: \beta > 1$

where β is the true slope of the linear relationship in Model 2.

Assume the sample data were independent (sufficiently randomized), the true relationship is linear, has a consistent standard deviation for any actual distance and is normally distributed in the residuals (y's)

t-test for slope of regression line

$$t = \frac{1.102 - 1}{0.393} = 0.2595$$



$df = 38$

$P(t > 0.2595 \mid H_0) = 0.3983$

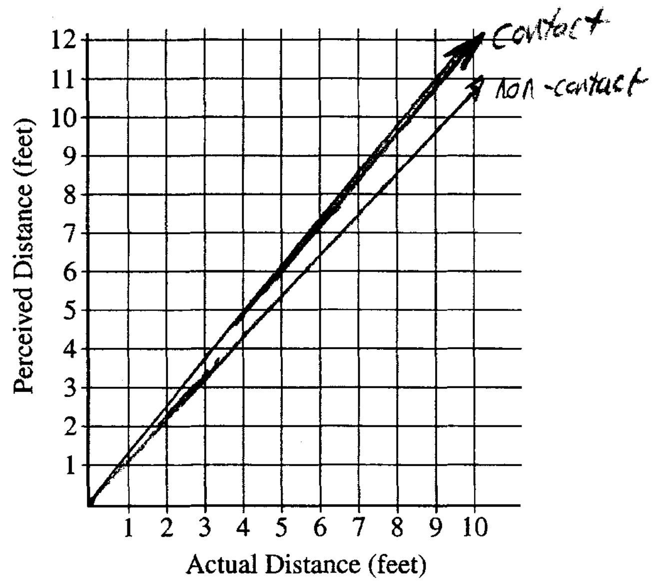
There is virtually no evidence that the researcher's hypothesis is correct, because if subjects were in fact unbiased perceivers of the distance, a result indicating at least this much of overestimating would occur nearly 40% of the time in any case.

The researchers also wanted to explore whether the performance on this task differed between subjects who wear contact lenses and subjects who do not wear contact lenses. A new variable was created to indicate whether or not a subject wears contact lenses. The data for this variable were coded numerically (1 = contact wearer, 0 = noncontact wearer), and this new variable, named "contact," was included in the following model.

Model 3: $\hat{y} = 1.05 \times (\text{actual distance}) + 0.12 \times (\text{contact}) \times (\text{actual distance})$

The standard errors of the estimated coefficients for Model 3 are 0.357 and 0.032, respectively.

(d) Using Model 3, sketch the estimated regression model for contact wearers and the estimated regression model for noncontact wearers on the grid below.



(e) In the context of this study, provide an interpretation of the estimated coefficients for Model 3.

Contact wearers overestimate the distance more.

model $\equiv$ $\hat{y}$ = (actual distance) + 0.05 × (actual distance) + 0.12 × (contact) × (actual distance)

Everyone overestimates by 5% of the actual distance on average; contact wearers overestimate by an additional 12% of the actual distance.

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   The standard errors of the estimated coefficients for Model 1 are 0.260 and 0.118, respectively.

   $$\text{Model 2: } \hat{y} = 1.102 \times (\text{actual distance})$$

   The standard error of the estimated coefficient for Model 2 is 0.393.

   (a) Provide an interpretation in context for the estimated slope in Model 1.

The slope in Model 1 suggests that the perceived distance between the objects increases by a factor of 1.080 (the slope) ft. for every 1 ft. increase in actual distance.

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(b) Explain why the researcher might prefer Model 2 to Model 1 in this context.

Model 2 might be preferred because the y-intercept is 0, suggesting that if two objects have a distance of 0 ft. btw. them, the respondent will most likely report a dist. of 0 ft. Model 1 gives a perceived distance of 0.238 ft. for a 0 ft. difference.

(c) Using Model 2, test the researcher's hypothesis that in dim light participants overestimate the distance, with the overestimate increasing as the actual distance increases. (Assume appropriate conditions for inference are met.)

① I'm interested in the actual slope of the linear model (2), or the factor by which the estimated difference increases for every foot increase in actual distance between 2 objects in a dimly lit room.

② I'll use a linear regression t-test for β. I assume that the perceived distances (y) are independent and that there is a uniform standard deviation around the linear regress. line.

If these conditions aren't met, I will proceed with caution.

③ $P(b \geq 1.102)$ $\quad P\left(t \geq \frac{1.102-0}{0.393}\right)$ $\quad P(t \geq 2.8041)$

$t = 2.8041$ $\quad df = n-2 = 40-2 = 38$ $\quad P = 0.0040$

④ Because my data is stat. sign. at the $\alpha = 0.05$ level, I reject that the actual slope is 0 (there is no correlation between actual and estimated distances btw. 2 objects in dim light) in favor that the slope is positive (perceived dist. increases as actual dist. increases).

$H_0: \beta = 0$ The slope is 0 – there is no correlation between actual and estimated distances btw. 2 objects in dim light.

$H_0: \beta > 0$ The slope is greater than 0 and the perceived distance increases as the actual distance increases.

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The researchers also wanted to explore whether the performance on this task differed between subjects who wear contact lenses and subjects who do not wear contact lenses. A new variable was created to indicate whether or not a subject wears contact lenses. The data for this variable were coded numerically (1 = contact wearer, 0 = noncontact wearer), and this new variable, named "contact," was included in the following model.

$$\text{Model 3: } \hat{y} = 1.05 \times (\text{actual distance}) + 0.12 \times (\text{contact}) \times (\text{actual distance})$$

The standard errors of the estimated coefficients for Model 3 are 0.357 and 0.032, respectively.

(d) Using Model 3, sketch the estimated regression model for contact wearers and the estimated regression model for noncontact wearers on the grid below.

contact wearer:

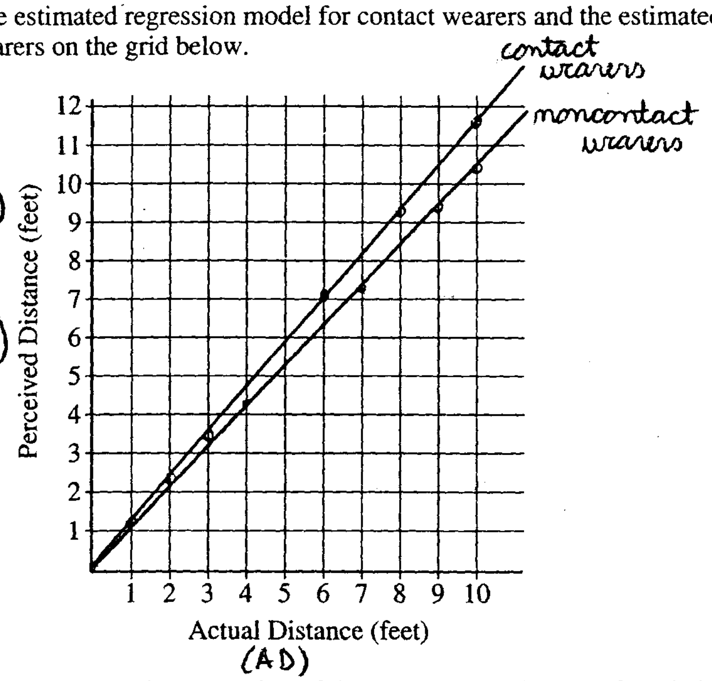
$\hat{y} = 1.05(AD) + 0.12(1)(AD)$

noncont. wearer:

$\hat{y} = 1.05(AD) + \cancel{0.12(0)(AD)}$

$\hat{y} = 1.05(AD)$

plug in AD values



(e) In the context of this study, provide an interpretation of the estimated coefficients for Model 3.

In model 3, the estimated coeff. 1.05 suggests that the perceived distance btw. 2 objects in a dim room increases by a factor of 1.05 for every 1 ft. increase in actual distance. This is hypothesized for both contact + noncontact wearers. For contact wearers, there is an extra increase by a factor of 0.12 (2nd coeff.) for every foot increase in AD.

The coeffs. imply that contact wearers overestimate the dist. in dim light more than noncontact wearers.

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   The standard errors of the estimated coefficients for Model 1 are 0.260 and 0.118, respectively.

   $$\text{Model 2: } \hat{y} = 1.102 \times (\text{actual distance})$$

   The standard error of the estimated coefficient for Model 2 is 0.393.

   (a) Provide an interpretation in context for the estimated slope in Model 1.

For every unit farther apart the objects are placed, the subject will estimate that the objects are an additional 1.080 units apart.

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6C2

(b) Explain why the researcher might prefer Model 2 to Model 1 in this context.

The researcher might prefer model 2 because the y-intercept is 0. It contains only one estimated value and therefore has less variability. The subject doesn't start off being automatically wrong when the researcher computes the expected values when the actual distance is zero.

(c) Using Model 2, test the researcher's hypothesis that in dim light participants overestimate the distance, with the overestimate increasing as the actual distance increases. (Assume appropriate conditions for inference are met.)

Linear Regression t-test $\beta \rightarrow$ the slope between the actual distance between objects, and the percived distance in dim light

$H_0: \beta = 1$: There is no difference between the actual distance and the percieved distance in dim light

$H_a: \beta > 1$: As the actual distance between the objects increases the distance percieved by the participant increases more.

All conditions for inference are met (given)

$b_h = 1$ hypothesized value

$b_2 = 1.102$ model2 value

$S_e = 0.393$ standard error

$\alpha = 0.05$

$$t = \frac{b_2 - b_h}{S_e} = \frac{1.102 - 1}{0.393} = 0.259$$

t-distribution

$t^* = 1.684$

$P(t^* > 1.684) = 0.05$

$P(t > 0.259) = 0.398 > 0.05$

Fail to reject H_0

There isn't sufficient evidence to reject H_0.

The slope between the actual distance & percieved distance is equal to 1. If H_0 were true, we would get results this extreme 39.8% of the time. This is not significant at the 0.05 level.

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6C3

The researchers also wanted to explore whether the performance on this task differed between subjects who wear contact lenses and subjects who do not wear contact lenses. A new variable was created to indicate whether or not a subject wears contact lenses. The data for this variable were coded numerically (1 = contact wearer, 0 = noncontact wearer), and this new variable, named "contact," was included in the following model.

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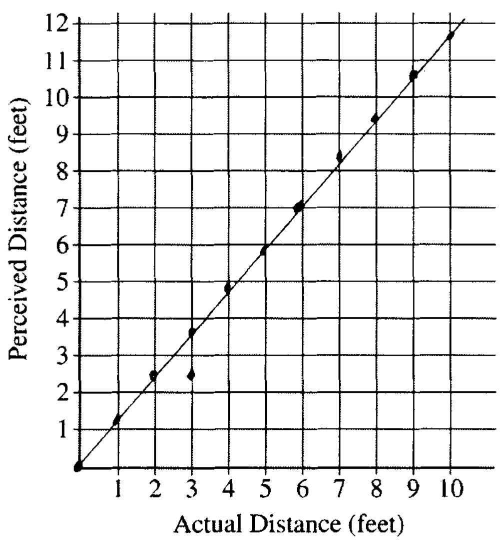

(e) In the context of this study, provide an interpretation of the estimated coefficients for Model 3.

Every participant overestimated the actual distance by 1.05 units. For every foot the actual distance increased, the perceived distance went up 1.05 feet. If someone wore contacts, their perceived distance went up an additional 0.12 feet on top of the 1.05 feet for every 1 foot increase in the actual distance.

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Question 6

Overview

This question was designed to evaluate a student's ability to make inferences for simple linear regression models. Interpreting model parameters and comparing and contrasting different models are important skills that are also being assessed. Finally, a multiple regression model with a special variable—an indicator variable—is introduced to investigate whether the relationship between the predictor and response variable differs for two different groups of people. Students are asked to sketch the estimated line for both groups and interpret the estimated parameters in the multiple regression model.

Sample: 6A
Score: 4

This outstanding response completely, concisely, and correctly answers all parts of this investigative task. Three insights were expected and are made: subjects are unlikely to perceive a distance greater than zero when the distance is zero, the researcher suspects that $\beta > 1$, and the effect of the indicator variable in Model 3 is to produce two lines. Part (a) gives a good interpretation of the estimated slope, making it clear that for every *additional* foot of actual distance, we *estimate* (or *predict*) that the subject will perceive 1.08 *additional* feet. Part (b) gives a correct explanation, but the term "directly proportional" would have been better than "directly linear." In part (c) the required parts of a test of significance are included: a statement of hypotheses, conditions, correct calculations, mechanics, and a conclusion that is based on the results of the computations. (In this particular test of significance, students are told that it is not necessary to state and check conditions.) The conclusion contains a good explanation of *p*-value, including the necessary qualifier that the *p*-value is computed assuming that the null hypothesis is true. The term "unbiased" in the conclusion to part (c) is used correctly. Using 38 degrees of freedom rather than 39 is considered a minor error. Parts (d) and (e) are concise and correct. This essay was complete in all essential ideas.

Sample: 6B
Score: 3

This response demonstrates a general understanding that subjects are unlikely to perceive a distance greater than zero when the distance is zero and that the effect of the indicator variable in Model 3 is to produce two lines. However, in part (c) the response makes the common error of testing the null hypothesis of $\beta = 0$. (A response that tested $\beta = 0$ in part (c) would not receive a score of 4.) Further, in the interpretation in part (a), 1.08 is referred to as a "factor." It is unclear if the response implies that the distance a person perceives can be found by multiplying the actual distance by 1.08, thereby ignoring the *y*-intercept and the uncertainty in the estimate. The difficulty about uncertainty also occurs in part (e), but the response was not penalized again for this. In part (c), while the hypotheses are incorrect, the calculations and conclusion are appropriate for the hypotheses stated. Using 38 degrees of freedom rather than 39 is considered a minor error. Parts (b), (d), and (e) were scored as correct. This is a substantial response.

Question 6 (continued)

Sample: 6C
Score: 2

The response demonstrates an understanding that the null hypothesis in part (c) should be $\beta = 1$ but does not demonstrate an understanding that Model 2 might be preferred nor does it demonstrate an understanding that the model with the indicator variable produces two linear equations. Further, in part (a) the statement “the subject will estimate” is too deterministic because an estimated slope of 1.080 does not imply that every person overestimates by the same amount or even that every person overestimates. The response would have been scored correct if the word “about” or “approximately” were used; for example, “the perceived distance between the objects increased by *approximately* 1.080 feet.” The same difficulty occurs in part (e), but the response was not penalized again for this. Part (b) was scored as incorrect. In fact, the estimated standard error for Model 2 is quite large compared to the two estimated standard errors for Model 1. Part (c) is very well done, with a good interpretation of the p-value. The wording of the null hypothesis implies that the null hypothesis is that each subject will predict correctly. A better wording would be, “For every increase of 1 foot in actual distance, on average people perceive an increase of 1 foot.” The single, unlabeled line in part (d) was scored as incorrect. Part (e) is a nice explanation of the estimated coefficients. The essay clearly illustrates a developing knowledge.