Question 2

Intent of Question

The primary goals of this question were to assess students' ability to (1) describe a sampling distribution of a sample mean; (2) set up and perform a normal probability calculation based on the sampling distribution.

Solution

Part (a):

The sampling distribution of the sample mean song length has mean $\mu_{\overline{X}} = \mu = 3.9$ minutes and standard deviation $\sigma_{\overline{X}} = \frac{\sigma}{\sqrt{n}} = \frac{1.1}{\sqrt{40}} \approx 0.174$ minutes. The central limit theorem (CLT) applies in this case because the sample size ($n = 40$) is fairly large, especially with the population of song lengths having a roughly symmetric distribution. Thus, the sampling distribution of the sample mean song length is approximately normal.

Part (b):

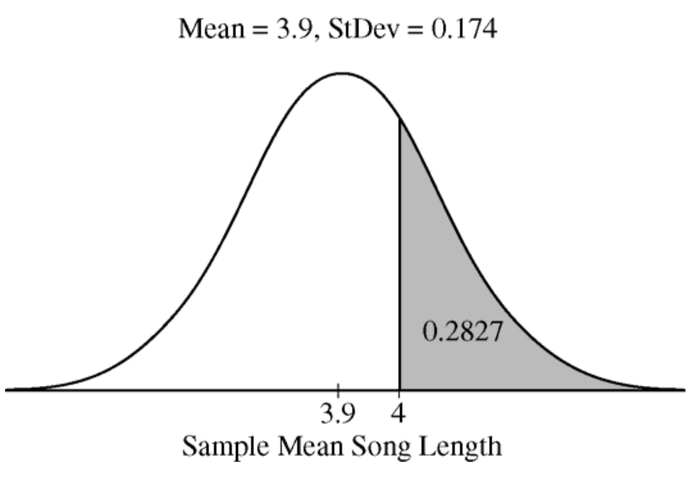
The probability that the total airtime of 40 randomly selected songs exceeds the available time (that is, the probability that the total airtime of 40 randomly selected songs is greater than 160 minutes) is equivalent to the probability that the sample mean length of the 40 songs is greater than $\frac{160}{40} = 4.0$ minutes.

According to part (a), the distribution of the sample mean length $\overline{X}$ is approximately normal. Therefore,

$$P\left(\overline{X} > 4.0\right) \approx P\left(Z > \frac{4.0 - 3.9}{0.174}\right) = P(Z > 0.57) = 1 - 0.7157 = 0.2843.$$

(The calculator gives the answer as 0.2827.)

The approximate sampling distribution of the sample mean song length and the desired probability are displayed below.



Question 2 (continued)

Part (b) (*alternative*):

An equivalent approach is to note that the sampling distribution of the total airtime, T, for the 40 songs is approximately normal, with mean $40(3.9) = 156$ minutes and standard deviation $\sqrt{40}(1.1) \approx 6.96$ minutes. The z-score for a total airtime of 160 minutes is then $z = \frac{160 - 156}{6.96} \approx 0.57$, and the calculation proceeds as above.

<u>Scoring</u>

Parts (a) and (b) are scored as essentially correct (E), partially correct (P) or incorrect (I).

Part (a) is scored as follows:

Essentially correct (E) if the student correctly provides all three components of the sampling distribution: shape (*approximately* normal), center (mean 3.9) and spread (standard deviation $\frac{1.1}{\sqrt{40}} \approx 0.174$).

Partially correct (P) if the student correctly provides only two of the three components.

Incorrect (I) if the student correctly provides only one or none of the components.

Notes

- Describing the sampling distribution as normal instead of approximately normal does not earn credit for the shape component.
- To earn credit for the spread component, the response must show how the standard deviation is calculated.
- If a response contains incorrect notation or terminology, it can at best be scored as partially correct (P).

Part (b) is scored as follows:

Essentially correct (E) if the student sets up and performs a correct normal probability calculation.

Partially correct (P) if the student sets up the normal probability calculation correctly but does not carry it through correctly *OR* sets up an incorrect but plausible calculation (for example, by using an incorrect standard deviation) but carries it through correctly.

Incorrect (I) if the student does not set up or perform the normal probability calculation correctly.

Question 2 (continued)

Notes

- A student can earn a score of essentially correct (E) in part (b) even with incorrect parameter values in part (a) by providing a correct calculation that uses the mean and standard deviation from part (a).
- Calculator syntax: An answer containing "normalcdf(…)" with no additional work or labeling is at best partially correct (P). If an appropriate sketch with the mean and standard deviation correctly labeled accompanies the calculator command, *OR* if the mean and standard deviation used in the calculator command are clearly identified in part (a) or part (b), then the response should be scored as essentially correct (E).
- If a student uses the sampling distribution of the total amount of time, T, needed to play the 40 randomly selected songs to do the probability calculation, the student must show how the standard deviation is calculated — unless this value is carried forward from part (a) — for the response to be scored as essentially correct (E). For example,

$$\sigma_T = \sqrt{40}\sigma_X = \sqrt{40}(1.1) \approx 6.96 \qquad \textit{OR} \qquad \sigma_T = 40\sigma_{\bar{X}} = 40(0.174) \approx 6.96.$$

4 Complete Response

Both parts essentially correct

3 Substantial Response

One part essentially correct and one part partially correct

2 Developing Response

One part essentially correct and one part incorrect

OR

Both parts partially correct

1 Minimal Response

One part partially correct and one part incorrect

2. A local radio station plays 40 rock-and-roll songs during each 4-hour show. The program director at the station needs to know the total amount of airtime for the 40 songs so that time can also be programmed during the show for news and advertisements. The distribution of the lengths of rock-and-roll songs, in minutes, is roughly symmetric with a mean length of 3.9 minutes and a standard deviation of 1.1 minutes. **2A**

(a) Describe the sampling distribution of the sample mean song lengths for random samples of 40 rock-and-roll songs.

The sampling distribution of the sample mean will be very nearly normal with a mean of 3.9 minutes and standard deviation of .17 minutes

$\mu_{\bar{x}} = \mu = 3.9 \text{ min}$

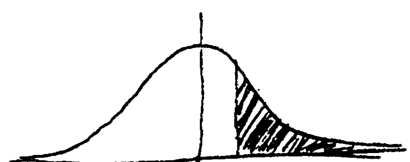
$\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}} = \frac{1.1 \text{ min}}{\sqrt{40}} = .17 \text{ min}$

(b) If the program manager schedules 80 minutes of news and advertisements for the 4-hour (240-minute) show, only 160 minutes are available for music. Approximately what is the probability that the total amount of time needed to play 40 randomly selected rock-and-roll songs exceeds the available airtime?

$\mu = n\mu_{\bar{x}} = 156$

$\sigma = n\sigma_{\bar{x}} = 6.96$

$z = \frac{160 - 156}{6.96} = .575$



$\boxed{P = .283}$

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2B

2. **A local radio station plays 40 rock-and-roll songs during each 4-hour show. The program director at the station needs to know the total amount of airtime for the 40 songs so that time can also be programmed during the show for news and advertisements. The distribution of the lengths of rock-and-roll songs, in minutes, is roughly symmetric with a mean length of 3.9 minutes and a standard deviation of 1.1 minutes.**

   (a) **Describe the sampling distribution of the sample mean song lengths for random samples of 40 rock-and-roll songs.**

Because of the sufficiently large sample size, the Central Limit Theorem shows that the sampling distribution of the sample mean song lengths for 40 rock-and-roll songs is approximately normal.

The mean of the distribution is equal to the population mean, or 3.9 minutes.

The standard deviation is equal to the population standard deviation divided by the square root of the sample size, or $\frac{1.1}{\sqrt{40}}$, approximately .174.

Therefore, the sampling distribution is $N(3.9, .174)$

   (b) **If the program manager schedules 80 minutes of news and advertisements for the 4-hour (240-minute) show, only 160 minutes are available for music. Approximately what is the probability that the total amount of time needed to play 40 randomly selected rock-and-roll songs exceeds the available airtime?**

Each song is about 3.9 minutes (see above); therefore, one can expect $40(3.9) = 156$ minutes for the mean song time of 40 rock-and-roll songs

The standard deviation can be expected to be $\sqrt{40(.174)^2}$, or 1.1

The sampling distribution of one song is approximately normal, so the shape of the distribution for the 40 songs should not be affected by the linear transformation – it should be approximately normal as well.

$P(\text{time to play 40 rock-and-roll songs} > 160) = P(z > \frac{160-156}{1.1})$

$\approx P(z > 3.636)$

$\approx 1.383 \times 10^{-4}$

There is an approximate 1.383×10^{-4} probability that the total amount of time needed to play 40 rock-and-roll songs exceeds the available airtime

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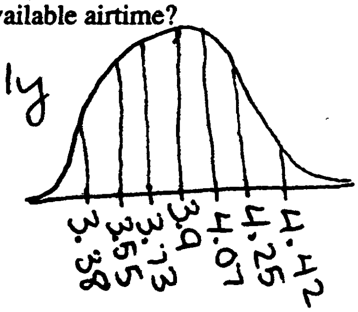
2. A local radio station plays 40 rock-and-roll songs during each 4-hour show. The program director at the station needs to know the total amount of airtime for the 40 songs so that time can also be programmed during the show for news and advertisements. The distribution of the lengths of rock-and-roll songs, in minutes, is roughly symmetric with a mean length of 3.9 minutes and a standard deviation of 1.1 minutes.

(a) Describe the sampling distribution of the sample mean song lengths for random samples of 40 rock-and-roll songs.

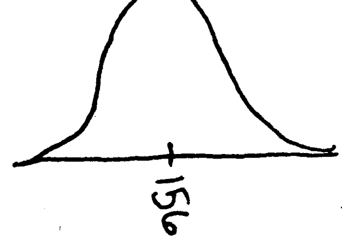
For a random sample of 40 rock-and-roll songs, the distribution would be centered at 3.9 minutes. Due to the Law of Diminishing Returns, the standard deviation of the sample would be $\frac{1.1}{\sqrt{40}}$, which is approximately .1739 minutes. The distribution would be roughly symmetrical and unimodal.

(b) If the program manager schedules 80 minutes of news and advertisements for the 4-hour (240-minute) show, only 160 minutes are available for music. Approximately what is the probability that the total amount of time needed to play 40 randomly selected rock-and-roll songs exceeds the available airtime?

The expected time to play 40 randomly selected rock-and-roll songs is 156 minutes. Therefore, the probability that the ~~necessary~~ necessary time to play 40 randomly selected rock-and-roll songs exceeds the available time is .284 or 28.4%.



$$z = \frac{160 - 156}{6.96} = .57$$



normal cdf(.57, 99) = .284

$$SD = 40\left(\frac{1.1}{\sqrt{40}}\right) \approx 6.96$$

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Question 2

Overview

The primary goals of this question were to assess students' ability to (1) describe a sampling distribution of a sample mean; (2) set up and perform a normal probability calculation based on the sampling distribution.

Sample: 2A
Score: 4

This is a very efficient, well-written response. The student describes all three components of the sampling distribution — shape, center and spread — correctly in part (a). In addition, the student provides an accurate justification (with good notation) for the values of the mean and standard deviation of this distribution. This part was scored as essentially correct. In part (b) the student begins by computing the mean and standard deviation of total airtime for random samples of 40 songs by using rules for linear transformation of a random variable. The student then sets up the z-score correctly and obtains the desired probability. This part was scored as essentially correct. Because parts (a) and (b) were both scored as essentially correct, the response earned a score of 4.

Sample: 2B
Score: 3

The response includes an accurate description with strong justification for the shape, center and spread of the sampling distribution in part (a). Although the student uses the notation "N(3.9, .174)" in the last line of the answer, the narrative comments clarify that the distribution "is approximately normal." This part was scored as essentially correct. The student correctly calculates the expected total airtime for 40 songs in part (b). However, the student's computation of the standard deviation is not correct. Using this incorrect standard deviation results in a plausible but incorrect z-score of 3.636. Because the student obtains the correct upper-tail probability that corresponds to this z-score, this part was scored as partially correct. With part (a) scored as essentially correct and part (b) scored as partially correct, the response earned a score of 3.

Sample: 2C
Score: 2

This response shows developing but incomplete understanding of sampling distributions. The first sentence in part (a) gives a correct value for the center ("3.9 minutes") but seems to be referring to the distribution of song lengths in a single random sample. The "standard deviation of the sample" remark in the second sentence confirms that the student is not describing the sampling distribution of the sample mean. As a result, this part was scored as incorrect. In part (b) the student gives the correct expected time required to play 40 randomly selected songs and computes the standard deviation of the total airtime correctly, by multiplying the values obtained in part (a) by 40. The student then sets up and performs the normal probability calculation correctly, so this part was scored as essentially correct. Because part (a) was scored as incorrect and part (b) was scored as essentially correct, the response earned a score of 2.