Question 4

Intent of Question

The primary goals of this question were to assess students' ability to (1) calculate and interpret a confidence interval for a population proportion; (2) recognize that it is still reasonable to use the confidence interval procedure even though sampling is without replacement as long as the sample size is small relative to the population size.

Solution

Part (a):

The sample proportion of songs that were loaded by Lori is $\hat{p} = \dfrac{13}{50} = 0.26$. The conditions for constructing a confidence interval are satisfied because: (1) the problem states that the 50 songs in the sample were randomly selected, and (2) $n \times \hat{p} = 13$ and $n \times (1 - \hat{p}) = 37$ are both at least 10. A 90 percent confidence interval for the population proportion p, the actual proportion of all songs on the player that were loaded by Lori, is:

$$0.26 \pm 1.645\sqrt{\frac{.26 \times (1 - .26)}{50}} = 0.26 \pm 0.102 = (0.158, 0.362).$$

We can be 90 percent confident that for the population of all songs on the digital music player, the proportion of songs that were loaded by Lori is between 0.158 and 0.362.

Part (b):

The sample size of 50 is quite small compared with the population size of 2,384. The usual criterion for checking whether one can disregard the distinction between sampling with or without replacement is to check whether the ratio of the population size to the sample size is large, such as at least 10 or at least 20. In this case the ratio is $\dfrac{2,384}{50} = 47.7$, so the criterion is clearly met, and the confidence interval procedure in part (a) is valid.

Scoring

This question is scored in four sections. Part (a) has three components: (1) stating the appropriate confidence interval procedure and checking its conditions; (2) construction of the confidence interval; (3) interpretation of the confidence interval. Section 1 consists of part (a), component 1; section 2 consists of part (a), component 2; section 3 consists of part (a), component 3. Section 4 consists of part (b). Each of the four sections is scored as essentially correct (E), partially correct (P) or incorrect (I).

Section 1 is scored as follows:

Essentially correct (E) if the response identifies a one-sample z-interval for a proportion (either by name or by formula) and also includes a statement of the random sampling condition and a statement of, and check of, the sample size condition.

Partially correct (P) if the response identifies the correct procedure but adequately addresses only one of the two conditions (random sampling, sample size) *OR* does not identify the correct procedure but adequately addresses both conditions.

Incorrect (I) if the response identifies the correct procedure but does not adequately address either condition *OR* does not identify the correct procedure and adequately addresses, at most, one condition.

Notes
- Stating only that "$n \times \hat{p}$ and $n \times (1 - \hat{p})$ are both greater than 10" is only a statement of the sample size condition and is not sufficient for checking it. The response must use specific values from the question in the check of the condition.
- If a response includes an inappropriate condition, such as requiring that $n \geq 30$ or requiring a normal population, then the response can earn no more than a P for part (a). However, stating and checking a condition about the size of the sample relative to the size of the population is not required but is also not inappropriate.

Section 2 is scored as follows:

Essentially correct (E) if the response makes use of the appropriate confidence interval procedure and calculates the 90 percent confidence interval correctly.

Partially correct (P) if the response makes use of the appropriate confidence interval procedure but does not include a correct calculation of the 90 percent confidence interval.

Incorrect (I) if the response makes use of an incorrect procedure, such as a *t*-interval for a population mean.

Section 3 is scored as follows:

Essentially correct (E) if the response provides a reasonable interpretation, in context, making clear that the estimate is for the population proportion of songs that were loaded by Lori and that we have 90 percent confidence in the interval.

Partially correct (P) if the response provides a reasonable interpretation, but does not make clear that the estimate is for the population proportion of songs that were loaded by Lori or does not mention 90 percent confidence.

Incorrect (I) if the response provides an incorrect interpretation.

Section 4 is scored as follows:

Essentially correct (E) if the response states that the difference between sampling with or without replacement is negligible here because the population size is large relative to the sample size *AND* provides a reasonable numerical justification for this assertion.

Partially correct (P) if the response states that the difference between sampling with or without replacement is negligible here because the population size is large relative to the sample size, but provides no numerical justification for this assertion.

Incorrect (I) if the response does not state that the difference between sampling with or without replacement is negligible here because the population size is large relative to the sample size.

Question 4 (continued)

Notes
- Reasonable numerical justification includes stating that the sample size is less than 5 percent (or 10 percent) of the population size or that the ratio of the population size to the sample size is greater than 20 (or 10).
- A response that compares the probabilities of songs being selected with or without replacement in terms of the sample and populations sizes, and that concludes that the difference in probabilities will be negligible, may be scored as essentially correct (E).

Each essentially correct (E) section counts as 1 point and each partially correct (P) section counts as ½ point.

4   **Complete Response**

3   **Substantial Response**

2   **Developing Response**

1   **Minimal Response**

If a response is between two scores (for example, 2½ points), use a holistic approach to determine whether to score up or down, depending on the strength of the response and communication.

4. A husband and wife, Mike and Lori, share a digital music player that has a feature that randomly selects which song to play. A total of 2,384 songs were loaded onto the player, some by Mike and the rest by Lori. Suppose that when the player was in the random-selection mode, 13 of the first 50 songs selected were songs loaded by Lori.

$$n = 2384$$

(a) Construct and interpret a 90 percent confidence interval for the proportion of songs on the player that were loaded by Lori.

50 songs were selected randomly to play.

13 songs were added by lori.

let p be the proportion of songs that was loaded by lori

Unbiased estimate of true p is $\hat{p} = \frac{13}{50} = 0.26$

Condition: 50 songs were selected randomly.

$$n\hat{p} = 50 \times 0.26 = 13 \ , \ n(1-\hat{p}) = 50(1-0.26) = 37$$

Since $n\hat{p} > 5$ and $p(1-\hat{p}) > 5$, we can assume that the sample size n is large enough to get a approximate normal distribution for $\hat{p}$

$$\sigma_{\hat{p}} = \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} = \sqrt{\frac{0.26(1-0.26)}{50}} = 0.0620032$$

Hence, we can construct z-interval

At 90 percent confidence interval, $\alpha = 0.10$, $z_{\frac{\alpha}{2}} = z_{\frac{0.10}{2}} = z_{0.05} = 1.6449$

$$ME = z_{\frac{\alpha}{2}}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} = 1.6449 \times 0.0620032 = 0.10204 = 0.102 \ (3dp)$$

$\hat{p} \pm ME = 0.26 \pm 0.102 \Rightarrow (0.158, 0.362)$ Hence, we are 90 percent confident that the true proportion of songs loaded by lori will be within 0.158 and 0.362.

(b) Mike and Lori are unsure about whether the player samples the songs with replacement or without replacement when the player is in random-selection mode. Explain why this distinction is not important for the construction of the interval in part (a).

This is because the number of songs is very large. Hence, the probability of choosing a song from 2384 songs with replacement and without replacement is approximately the same.

For example, the probability of choosing the second song:

- with replacement $= \frac{1}{2384} = 4.1946 \times 10^{-4}$.

- without replacement $= \frac{1}{2384-1} = 4.1964 \times 10^{-4}$.

The difference is only 1.76×10^{-7} which is so small that can be ignored.

4. A husband and wife, Mike and Lori, share a digital music player that has a feature that randomly selects which song to play. A total of 2,384 songs were loaded onto the player, some by Mike and the rest by Lori. Suppose that when the player was in the random-selection mode, 13 of the first 50 songs selected were songs loaded by Lori.

(a) Construct and interpret a 90 percent confidence interval for the proportion of songs on the player that were loaded by Lori.

This is one-sample-proportion-confidence interval.

The assumption is checked: ① The selection is randomly

② $\frac{13}{50} \times 50 = 13 > 10$; $\frac{37}{50} \times 50 = 37 > 10$

③ 50 is less than 10% of population 238.4

The 90 percent confidence :

$$\hat{p} = \frac{13}{50} = 0.26$$

$$\hat{p} \pm 1.64 \cdot \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} = 0.26 \pm 1.64 \cdot \sqrt{\frac{0.26 \cdot 0.74}{50}} = 0.26 \pm 0.10$$

$$(0.16 , 0.36)$$

So, we are 90% confident that the proportion of songs on the player that were loaded by Lori is between 0.16 and 0.36

(b) Mike and Lori are unsure about whether the player samples the songs with replacement or without replacement when the player is in random-selection mode. Explain why this distinction is not important for the construction of the interval in part (a).

Because the assumption or requirement for confidence interval is only $\begin{cases} \text{normally distributed or with no outliers.} \\ \text{randomly selected.} \\ \text{independent.} \end{cases}$ (These are all attained).

So, the replacement or not will not affect the result in part (a).

GO ON TO THE NEXT PAGE.

-12-

4. A husband and wife, Mike and Lori, share a digital music player that has a feature that randomly selects which song to play. A total of 2,384 songs were loaded onto the player, some by Mike and the rest by Lori. Suppose that when the player was in the random-selection mode, 13 of the first 50 songs selected were songs loaded by Lori.

(a) Construct and interpret a 90 percent confidence interval for the proportion of songs on the player that were loaded by Lori.

$$p = \frac{13}{50} = 0.26 \qquad \sigma = \sqrt{\frac{0.26 \times (1-0.26)}{50}} \approx 0.06$$

$$\text{a} \quad \hat{p} = 0.26 \pm 1.64 \times 0.06$$

So ~~we are 90% confident that~~ $(0.1616, 0.3584)$ is the 90 percent confidence interval for the proportion of songs on the player that were loaded by Lori.

(b) Mike and Lori are unsure about whether the player samples the songs with replacement or without replacement when the player is in random-selection mode. Explain why this distinction is not important for the construction of the interval in part (a).

If with replacement:
the probability for a song to be selected is $p = \frac{1}{2384} \approx 0.00042$

Without a replacement:

~~the p to~~
As the probability for a song to be selected will become larger as the process goes on, and the largest probability

$p = \frac{1}{2335} \approx \cancel{4 \times 10^{-4}} \quad 0.00043$

So the difference between the two probabilities is small enough to be ignored. So this distinction is ~~also~~ not important for the construction of the interval in part (a)

-12-

Question 4

Sample: 4A
Score: 4

In part (a) the student identifies the appropriate confidence interval procedure by listing its components before constructing the interval. The student states and checks the conditions for inference, computes the confidence interval, and provides a correct interpretation of the interval in the specific context of the proportion of songs loaded by Lori. In part (b) the student presents an argument illustrating that the change in the probability of a specific song's being selected is negligible if a different song is selected first without replacement. Section 1, consisting of part (a), component 1; section 2, consisting of part (a), component 2; section 3, consisting of part (a), component 3; and section 4, consisting of part (b), were each scored as essentially correct. The entire answer, based on all four sections, was judged a complete response and earned a score of 4.

Sample: 4B
Score: 3

In part (a) the student identifies the appropriate confidence interval by stating the formula, states and checks conditions, computes the interval, and interprets it correctly. Section 1, consisting of part (a), component 1; section 2, consisting of part (a), component 2; and section 3, consisting of part (a), component 3, were each scored as essentially correct. The response to part (b) is focused not on the relationship between the size of the sample and the size of the population but instead on the other conditions for inference associated with the confidence interval procedure. Section 4, consisting of part (b), was scored as incorrect. The entire answer, based on all four sections, was judged a substantial response and earned a score of 3.

Sample: 4C
Score: 2

In part (a) the student identifies the correct procedure by the formula shown, but the response does not include a statement or check of the conditions for inference. Section 1, consisting of part (a), component 1, received a score of incorrect. The student then uses an appropriate confidence interval procedure and correctly computes the 90 percent confidence interval. Section 2, consisting of part (a), component 2, received a score of essentially correct. The conclusion in part (a) states only that this is the confidence interval and is not a correct interpretation of what the interval means. Section 3, consisting of part (a), component 3, received a score of incorrect. Part (b) contains an insightful argument for why the difference between sampling with or without replacement is negligible for a population and a sample of this size. Section 4, consisting of part (b), received a score of essentially correct. The entire answer, based on all four sections, was judged a developing response and earned a score of 2.