Question 6

Intent of Question

The primary goals of this question were to assess students' ability to (1) implement simple random sampling; (2) calculate an estimated standard deviation for a sample mean; (3) use properties of variances to determine the estimated standard deviation for an estimator; (4) explain why stratification reduces a standard error in a particular study.

Solution

Part (a):

Peter can number the students from 1 to 2,000 and then use a calculator or computer to generate 100 unique random numbers between 1 and 2,000 without replacement. If non-unique numbers are generated, the repeated numbers are ignored until 100 unique numbers are obtained. The students whose numbers correspond to the randomly generated numbers are then selected for the sample.

Part (b):

The estimated standard deviation of the sampling distribution of the sample mean is:
$\frac{s}{\sqrt{n}}$, or $\frac{4.13}{\sqrt{100}} = 0.413$.

Part (c):

The variance of Rania's estimator is $(0.6)^2 \text{Var}\left(\bar{X}_f\right) + (0.4)^2 \text{Var}\left(\bar{X}_m\right)$, where $\text{Var}\left(\bar{X}_f\right) = \frac{\sigma_f^2}{n_f}$ represents the variance of the point estimator for females and $\text{Var}\left(\bar{X}_m\right) = \frac{\sigma_m^2}{n_m}$ represents the variance of the point estimator for males.

The estimated standard deviation is the square root of the variance. Using the respective sample standard deviations s_f and s_m for the population parameters, Rania's estimate is calculated as:

$$\sqrt{(0.6)^2 \frac{s_f^2}{n_f} + (0.4)^2 \frac{s_m^2}{n_m}} = \sqrt{(0.6)^2 \frac{(1.80)^2}{60} + (0.4)^2 \frac{(2.22)^2}{40}} \approx \sqrt{0.01944 + 0.01972} \approx 0.198.$$

Part (d):

The comparative dotplots from Rania's data reveal that the distribution of the number of soft drinks for females appears to be quite different from that of males. In particular, the centers of the distributions appear to be significantly different. Additionally, the variability of values around the center *within* gender in each of Rania's dotplots appears to be considerably less than the variability displayed in the dotplot of Peter's data. Rania's estimator takes advantage of the decreased variability within gender because her data were obtained by sampling the two genders separately. Peter's estimator has more variability because his data were obtained from a simple random sample of all the high school students.

Question 6 (continued)

<u>Scoring</u>

Parts (a), (b), (c), and (d) are scored as essentially correct (E), partially correct (P), or incorrect (I).

Part (a) is scored as follows:

Essentially correct (E) if the response provides a correct sampling procedure to obtain a simple random sample that includes sufficient information such that two knowledgeable statistics users would implement equivalent methods.

Partially correct (P) if the response provides a plausible sampling procedure for obtaining a simple random sample but does *NOT* include sufficient information as described for E.

Incorrect (I) if the response does not meet the criteria for E or P.

Notes

- If computer-generated random numbers are used, an explicit statement about ignoring repeats is needed for sufficient information. (Sampling until 100 students are obtained conveys the idea of ignoring repeats.)
- If a table of random numbers is used without adequate specification of the numbers to be used (for example, 0001 to 2,000 or a different set of 2,000 four-digit numbers), the response is scored as incorrect (I).
- If a table of random numbers is used with adequate specification of the numbers to be used, as described above, a statement about ignoring repeats *AND* a statement about ignoring numbers not in the specified range are both needed to earn an E. If either or both are missing, the response is scored as partially correct (P). (The statement about ignoring repeats or the statement about ignoring numbers not in the range of specification may be implicit. For example, "… continue this procedure until 100 students are selected.")
- The procedure of using names or numbers on slips of paper in a hat must indicate some randomization in selection (for example, mixing the slips of papers or randomly choosing from the hat).
- If the sampling procedure does not produce a simple random sample, then the response is scored as incorrect (I).

Part (b) is scored as follows:

Essentially correct (E) if the correct calculation is performed *AND* sufficient work is shown.

Partially correct (P) if the correct answer is provided but no work is shown.

Incorrect (I) if the response does not meet the criteria for E or P.

Notes

- The correct formula (either in symbols or with correct numerical substitutions) is sufficient for shown work in either part (b) or part (c).
- A response with only the answer 0.413 with no work shown is scored as P (because dividing by the square root of 100 can be done mentally).

Question 6 (continued)

- Sufficient shown work with no calculation is scored as partially correct (P). For example, showing $\frac{4.13}{\sqrt{100}}$ with no calculation.
- Minor calculation errors can be ignored in part (b) or part (c) and considered when determining between two scores.
- The incorrect use of the notation σ instead of s is considered a minor error in the investigative task and will not reduce the score in either part (b) or part (c).

Part (c) is scored as follows:

Essentially correct (E) if the response includes the following three components:

1. Combining variances.
2. Using weights (0.6 for females and 0.4 for males).
3. Recognizing variability of sample means (variance divided by sample size),

AND

the response correctly combines these three components to produce the estimated standard deviation for Rania's point estimator.

Partially correct (P) if the response has at least two of the three components *AND* a reasonable attempt to combine these components to produce the (overall) estimated standard deviation for Rania's point estimator.

Incorrect (I) if the response does not meet the criteria for E or P.

Notes

- When part (c) is scored partially correct (P), the reasonableness of the numerical result can be considered in determining the overall score.
- The following are algebraically equal and provide the correct estimated standard deviation of Rania's point estimator of 0.198:

$$\sqrt{(0.6)^2\frac{s_f^2}{n_f}+(0.4)^2\frac{s_m^2}{n_m}}=\sqrt{(0.6)^2\frac{s_f^2}{60}+(0.4)^2\frac{s_m^2}{40}}=\sqrt{(0.6)\frac{s_f^2}{100}+(0.4)\frac{s_m^2}{100}}=\sqrt{\frac{(0.6)s_f^2+(0.4)s_m^2}{100}}$$

- A response that combines s_f^2 and s_m^2 to obtain a pooled standard deviation for Rania's data, such as

$$s_p=\sqrt{\frac{(n_f-1)s_f^2+(n_m-1)s_m^2}{(n_f-1)+(n_m-1)}}=\sqrt{\frac{(59)(1.8)^2+(39)(2.22)^2}{59+39}}\approx 1.97786,$$

contains two of the three components: combining variances and using weights $\left(\frac{59}{98}\approx 0.6 \text{ and } \frac{39}{98}\approx 0.4\right)$ for part (c). Because pooling is a reasonable attempt to combine the components, the calculation of $s_p \approx 1.97786$ is scored as partially correct (P). If the response includes the correct third component, $\left(\frac{s_p}{\sqrt{n}}\approx\frac{1.97786}{\sqrt{100}}=0.197786\right)$, the response is still scored partially correct, because the pooling is a reasonable, but not the correct, combination of the three components to compute the sample standard deviation of the stratified sample mean.

Question 6 (continued)

Part (d) is scored as follows:

Essentially correct (E) if a reasonable justification is provided based on *BOTH* of the following two components:

1. Smaller variability in responses for each gender in comparison with the variability in responses in Peter's data, and
2. Linkage (either explicit or implicit) between the smaller variability in responses for each gender and producing a smaller estimated standard deviation for Rania's combined point estimator than Peter's point estimator.

Partially correct (P) if the benefit of the smaller variability in responses for each gender with stratification (homogeneity within each gender) for these samples is identified, but there is no reference to how it affects the estimated standard error for Rania's point estimator,

OR

if the benefit of different centers with stratification (heterogeneity between genders) for the samples is identified, without reference to the resulting smaller variability in responses for each gender and how it affects the estimated standard error for Rania's point estimator,

OR

if the response identifies that Rania's combined data for males and females has smaller variability than Peter's data.

Incorrect (I) if the response does not meet the criteria for E or P.

Notes

- Comments about shapes of the distributions are extraneous and can be ignored.
- General statements about the benefits of stratification without using the sample data in the dotplots are not sufficient.

Each essentially correct (E) part counts as 1 point. Each partially correct (P) part counts as ½ point.

4 **Complete Response**

3 **Substantial Response**

2 **Developing Response**

1 **Minimal Response**

If a response is between two scores (for example, 2½ points), use a holistic approach to decide whether to score up or down, depending on the overall strength of the response and communication, especially in the investigative parts—part (c) and part (d)—of the response.

6A1 6A1

STATISTICS
SECTION II
Part B
Question 6
Spend about 25 minutes on this part of the exam.
Percent of Section II score—25

Directions: Show all your work. Indicate clearly the methods you use, because you will be scored on the correctness of your methods as well as on the accuracy and completeness of your results and explanations.

6. Two students at a large high school, Peter and Rania, wanted to estimate μ, the mean number of soft drinks that a student at their school consumes in a week. A complete roster of the names and genders for the 2,000 students at their school was available. Peter selected a simple random sample of 100 students. Rania, knowing that 60 percent of the students at the school are female, selected a simple random sample of 60 females and an independent simple random sample of 40 males. Both asked all of the students in their samples how many soft drinks they typically consume in a week.

(a) Describe a method Peter could have used to select a simple random sample of 100 students from the school.

For Peter to select a simple random sample of 100 students from the school, he could get a list of all 2000 students that attend the school. Number the students from 0000 to 1999. use a random integers table, starting from the first line, move across the table four digits at a time. Record the number skip any number over 1999, and ignore repeats. once the first 100 numbers are selected the students corresponding to those numbers will be the 100 students of the sample.

GO ON TO THE NEXT PAGE.

6A2 6A2

Peter and Rania conducted their studies as described. Peter used the sample mean $\overline{X}$ as a point estimator for μ. Rania used $\overline{X}_{\text{overall}} = (0.6)\overline{X}_{\text{female}} + (0.4)\overline{X}_{\text{male}}$ as a point estimator for μ, where $\overline{X}_{\text{female}}$ is the mean of the sample of 60 females and $\overline{X}_{\text{male}}$ is the mean of the sample of 40 males.

Summary statistics for Peter's data are shown in the table below.

Variable	N	Mean	Standard Deviation
Number of soft drinks	100	5.32	4.13

(b) Based on the summary statistics, calculate the estimated standard deviation of the sampling distribution (sometimes called the standard error) of Peter's point estimator $\overline{X}$.

$$\frac{4.13}{\sqrt{100}} = .413$$

Summary statistics for Rania's data are shown in the table below.

Variable	Gender	N	Mean	Standard Deviation
Number of soft drinks	Female	60	2.90	1.80
	Male	40	7.45	2.22

(c) Based on the summary statistics, calculate the estimated standard deviation of the sampling distribution of Rania's point estimator $\overline{X}_{\text{overall}} = (0.6)\overline{X}_{\text{female}} + (0.4)\overline{X}_{\text{male}}$.

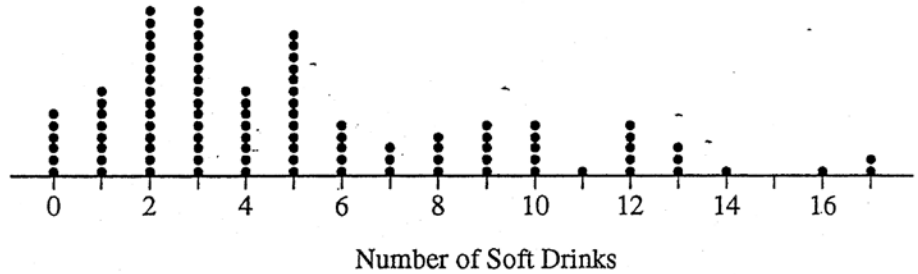
$$\sqrt{(.6)^2\left(\frac{1.8}{\sqrt{60}}\right)^2 + (.4)^2\left(\frac{2.22}{\sqrt{40}}\right)^2} = .198$$

GO ON TO THE NEXT PAGE.

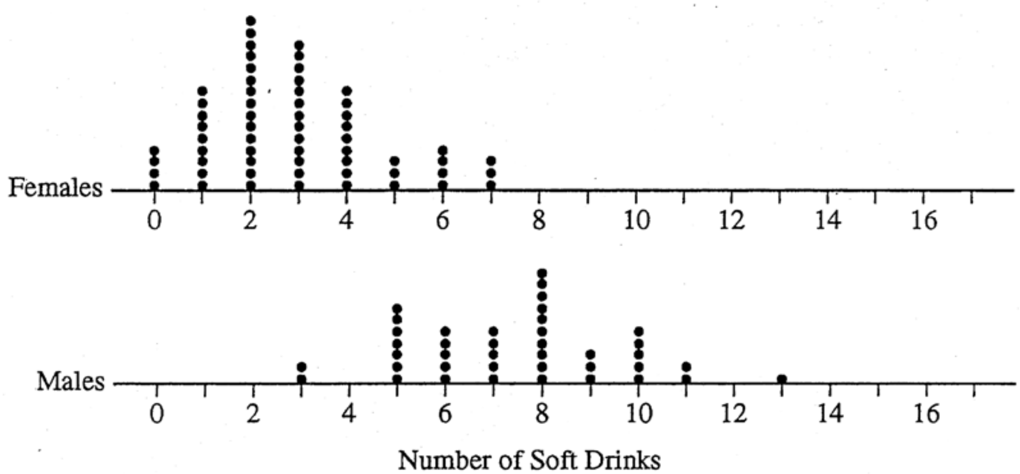
6A3

6A3

A dotplot of Peter's sample data is given below.



Comparative dotplots of Rania's sample data are given below.



(d) Using the dotplots above, explain why Rania's point estimator has a smaller estimated standard deviation than the estimated standard deviation of Peter's point estimator.

Rania's point estimator has a smaller estimated standard deviation than the estimated standard deviation of Peter's point estimator because Rania used two independent random samples. Each random sample had a different standard deviation and sample size, both which are smaller than Peter's sample size and standard deviation. Because of the rule for variances needed to calculate Rania's overall sample standard deviation, the small values would result in small estimated standard deviation.

GO ON TO THE NEXT PAGE.

6B1 6B1

STATISTICS
SECTION II
Part B
Question 6
Spend about 25 minutes on this part of the exam.
Percent of Section II score—25

Directions: Show all your work. Indicate clearly the methods you use, because you will be scored on the correctness of your methods as well as on the accuracy and completeness of your results and explanations.

6. Two students at a large high school, Peter and Rania, wanted to estimate μ, the mean number of soft drinks that a student at their school consumes in a week. A complete roster of the names and genders for the 2,000 students at their school was available. Peter selected a simple random sample of 100 students. Rania, knowing that 60 percent of the students at the school are female, selected a simple random sample of 60 females and an independent simple random sample of 40 males. Both asked all of the students in their samples how many soft drinks they typically consume in a week.

(a) Describe a method Peter could have used to select a simple random sample of 100 students from the school.

Peter can use a random number generator from 1 to 2000. Peter will generate a random number each time and the student's name that corresponds to the number is selected. Peter will do this 100 times. If he gets a repeated number, he will redo the number generator for a new number.

GO ON TO THE NEXT PAGE.

Peter and Rania conducted their studies as described. Peter used the sample mean $\overline{X}$ as a point estimator for μ. Rania used $\overline{X}_{\text{overall}} = (0.6)\overline{X}_{\text{female}} + (0.4)\overline{X}_{\text{male}}$ as a point estimator for μ, where $\overline{X}_{\text{female}}$ is the mean of the sample of 60 females and $\overline{X}_{\text{male}}$ is the mean of the sample of 40 males.

Summary statistics for Peter's data are shown in the table below.

Variable	N	Mean	Standard Deviation
Number of soft drinks	100	5.32	4.13

(b) Based on the summary statistics, calculate the estimated standard deviation of the sampling distribution (sometimes called the standard error) of Peter's point estimator $\overline{X}$.

$$\frac{4.13}{\sqrt{100}} = \boxed{0.413}$$

Summary statistics for Rania's data are shown in the table below.

Variable	Gender	N	Mean	Standard Deviation
Number of soft drinks	Female	60	2.90	1.80
	Male	40	7.45	2.22

(c) Based on the summary statistics, calculate the estimated standard deviation of the sampling distribution of Rania's point estimator $\overline{X}_{\text{overall}} = (0.6)\overline{X}_{\text{female}} + (0.4)\overline{X}_{\text{male}}$.

$$\frac{1.80}{\sqrt{60}} = 0.232$$

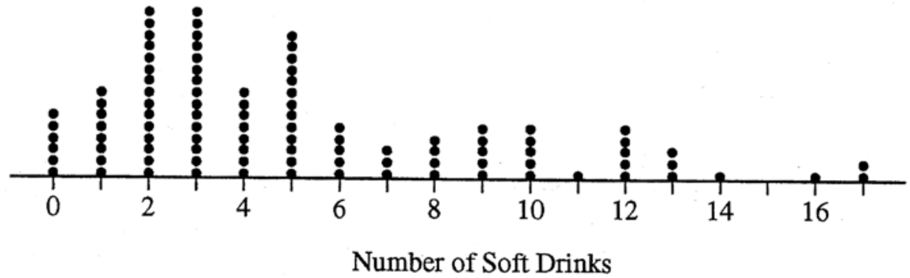
$$\frac{2.22}{\sqrt{40}} = 0.351$$

$$0.6(0.232) + 0.4(0.351)$$

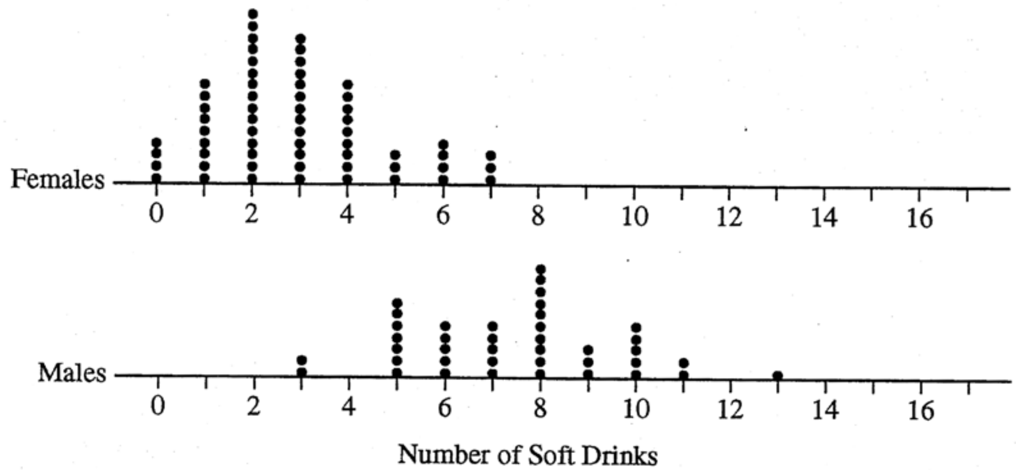
$$= \boxed{0.280}$$

GO ON TO THE NEXT PAGE.

 A dotplot of Peter's sample data is given below. 6B3



Comparative dotplots of Rania's sample data are given below.



(d) Using the dotplots above, explain why Rania's point estimator has a smaller estimated standard deviation than the estimated standard deviation of Peter's point estimator.

From the dotplots, females tend to drink less soft drinks per week than males. Since Rania used a sample that was 60% female and 40% male, her standard deviation should be smaller than Peter's standard deviation, who used a simple random sample of 100 students regardless of gender.

GO ON TO THE NEXT PAGE.

6C1

6C1

STATISTICS
SECTION II
Part B
Question 6
Spend about 25 minutes on this part of the exam.
Percent of Section II score—25

Directions: Show all your work. Indicate clearly the methods you use, because you will be scored on the correctness of your methods as well as on the accuracy and completeness of your results and explanations.

6. Two students at a large high school, Peter and Rania, wanted to estimate μ, the mean number of soft drinks that a student at their school consumes in a week. A complete roster of the names and genders for the 2,000 students at their school was available. Peter selected a simple random sample of 100 students. Rania, knowing that 60 percent of the students at the school are female, selected a simple random sample of 60 females and an independent simple random sample of 40 males. Both asked all of the students in their samples how many soft drinks they typically consume in a week.

(a) Describe a method Peter could have used to select a simple random sample of 100 students from the school.

Peter could have randomly assigned each name on the roster a number from 1 to 2000. He could have then chosen every multiple of 20 (starting with person #20) on the list.

Peter could also put the names of each of the 2000 students into a hat, mixed up the names in the hat, and then draw out 100 names. This would ensure random sampling.

GO ON TO THE NEXT PAGE.

Peter and Rania conducted their studies as described. Peter used the sample mean $\overline{X}$ as a point estimator for μ. Rania used $\overline{X}_{\text{overall}} = (0.6)\overline{X}_{\text{female}} + (0.4)\overline{X}_{\text{male}}$ as a point estimator for μ, where $\overline{X}_{\text{female}}$ is the mean of the sample of 60 females and $\overline{X}_{\text{male}}$ is the mean of the sample of 40 males.

Summary statistics for Peter's data are shown in the table below.

Variable	N	Mean	Standard Deviation
Number of soft drinks	100	5.32	4.13

(b) Based on the summary statistics, calculate the estimated standard deviation of the sampling distribution (sometimes called the standard error) of Peter's point estimator $\overline{X}$.

$$SE = \frac{s}{\sqrt{n}} = \frac{4.13}{\sqrt{100}} = \frac{4.13}{10} = .413$$

Summary statistics for Rania's data are shown in the table below.

Variable	Gender	N	Mean	Standard Deviation
Number of soft drinks	Female	60	2.90	1.80
	Male	40	7.45	2.22

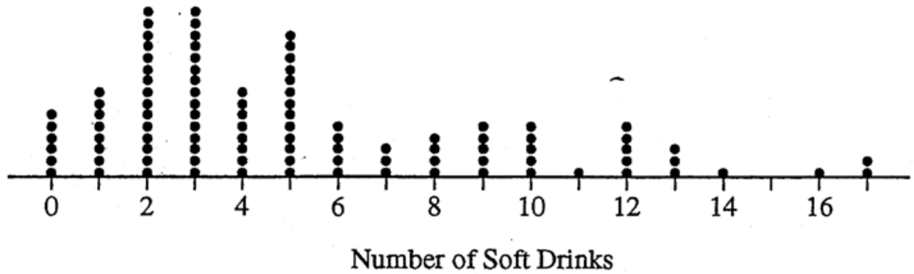
(c) Based on the summary statistics, calculate the estimated standard deviation of the sampling distribution of Rania's point estimator $\overline{X}_{\text{overall}} = (0.6)\overline{X}_{\text{female}} + (0.4)\overline{X}_{\text{male}}$.

$$SE_F = \frac{s_F}{\sqrt{n_F}} = \frac{1.8}{\sqrt{60}} = .23 \text{ female}$$

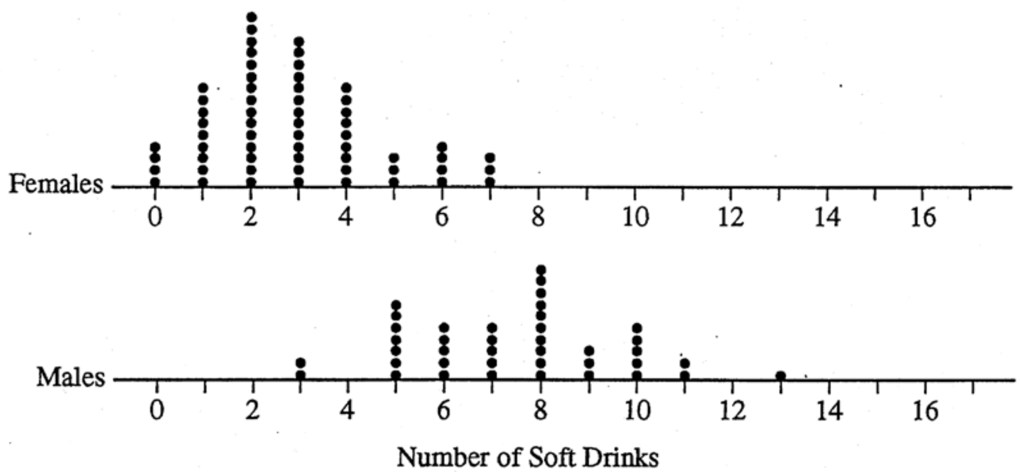
$$SE_M = \frac{s_m}{\sqrt{n_m}} = \frac{2.22}{\sqrt{40}} = .35 \text{ male}$$

6C3 6C3

A dotplot of Peter's sample data is given below.



Comparative dotplots of Rania's sample data are given below.



(d) Using the dotplots above, explain why Rania's point estimator has a smaller estimated standard deviation than the estimated standard deviation of Peter's point estimator.

Raina's point estimator has a smaller estimated standard deviation than Peter's point estimator because she drew her sample differently. Rainā took into account that 60% (majority) of the students in her school were female ~~so~~ her data accomodates that. Peter, on the other hand, claims his sample represents the school's population ~~when~~, in reality, the # of females affected his data. Even in the dotplots, you can see the breakdown of how females drink a significantly fewer soft drinks. Clearly, Rania also has a smaller sample size (fewer dots) in her grouping (smaller n) which leads to a smaller standard of error

GO ON TO THE NEXT PAGE.

Question 6

Overview

The primary goals of this question were to assess students' ability to (1) implement simple random sampling; (2) calculate an estimated standard deviation for a sample mean; (3) use properties of variances to determine the estimated standard deviation for an estimator; (4) explain why stratification reduces a standard error in a particular study.

Sample: 6A
Score: 4

In part (a) the 2,000 students in the high school are assigned numbers from 0000 to 1,999. A random selection of 100 unique numbers from a random integers table is clearly described by ignoring repeated four-digit numbers and by ignoring the four-digit numbers greater than 1,999. The 100 unique random numbers are used to select the 100 students with the corresponding numbers. Thus, part (a) was scored as essentially correct. In part (b) the formula for the standard deviation of the sample mean is correctly identified by the appropriate statistics from the sample, and these statistics are used to calculate the correct standard error of Peter's point estimator. Thus, part (b) was scored as essentially correct. The variances, weights, and sample sizes for female responses and male responses in Rania's samples are correctly combined in part (c) to produce the correct estimated standard deviation of the stratified sample mean. Thus, part (c) was scored as essentially correct. In part (d) the standard deviations for each gender are identified as being smaller in Rania's sample data than the standard deviation in Peter's sample data. The two smaller variances for each gender are explicitly linked to the smaller sample standard deviation for Rania's point estimator with the phrase "the rule for variances needed to calculate Rania's." Hence, part (d) was scored as essentially correct. With all four parts scored as essentially correct, the response earned a score of 4.

Sample: 6B
Score: 3

In part (a) the response implicitly identifies each student with a number from 1 to 2,000. A random number generator is used to produce 100 unique numbers, and the students corresponding to the 100 unique numbers are selected for the simple random sample. Thus, part (a) was scored as essentially correct. In part (b) the formula for the standard deviation of the sample mean is correctly identified by use of the appropriate sample statistics, and the correct standard error of Peter's point estimator is calculated. Thus, part (b) was scored as essentially correct. In part (c) the standard errors for the sample means for the female responses and male responses are correctly calculated, but the attempt to combine the standard deviations rather than the variances produces an incorrect estimate for the standard deviation of Rania's point estimator. Thus, part (c) was scored as partially correct. The response in part (d) identifies the difference in sample means for the female and male responses, but the linkage of the reduced variability in the stratified sample to a smaller estimated standard deviation of the estimator for the mean is insufficient. Thus, part (d) was scored as partially correct. With two parts scored as essentially correct and two parts scored as partially correct, the response earned a score of 3.

Question 6 (continued)

Sample: 6C
Score: 2

There are parallel solutions in the response for part (a). First, each name is randomly assigned a number from 1 to 2,000. Then 100 students are selected by choosing every 20th person on the randomized list. Because the list is randomized, any selection of 100 students results in a simple random sample. However, insufficient information is provided on the randomization of the list, so the solution was scored as partially correct. The second procedure of putting the names of the 2,000 students into a hat and randomly selecting 100 names from the hat is an essentially correct response. However, considering both responses, part (a) was scored as partially correct. In part (b) the formula for the standard deviation of the sample mean is correctly stated, and the appropriate sample statistics are used to calculate the correct standard error of Peter's point estimator. Thus, part (b) was scored as essentially correct. In part (c) the standard errors for the sample means for the female responses and male responses are correctly calculated, but there is no attempt to combine the quantities to produce a single estimate for the standard deviation of Rania's point estimator. Thus, part (c) was scored as incorrect. The response in part (d) identifies the difference in means for the female and male responses (heterogeneity between genders) but does not identify the variability for each gender being smaller in Rania's sample data than the variability in Peter's sample data. Moreover, the smaller sample sizes do not necessarily lead to smaller standard errors. Thus, part (d) was scored as partially correct. With one part scored as essentially correct, two parts scored as partially correct, and one part scored as incorrect, the response earned a score of 2.