Question 3

Intent of Question

The primary goals of this question were to assess a student's ability to (1) calculate a probability from a normal distribution and (2) apply properties of means and variances of functions of random variables.

Solution

Part (a):

Let W denote the weight of a randomly selected full carton of eggs. W has a normal distribution with mean 840 grams and standard deviation 7.9 grams.

The z-score for a weight of 850 grams is $z = \dfrac{850 - 840}{7.9} \approx 1.27.$

The standard normal probability table reveals that
$P(W > 850) = P(Z > 1.27) \approx 1 - 0.8980 = 0.1020.$

Part (b):

(i) Let W represent the weight of a randomly selected full carton of eggs, P the weight of the packaging, and X_i the weight of the *ith* egg, for $i = 1, 2, \ldots, 12.$

Note that $W = P + X_1 + X_2 + \ldots + X_{12}.$
Properties of expected values establish that $\mathrm{E}(W) = \mathrm{E}(P) + \mathrm{E}(X_1) + \ldots + \mathrm{E}(X_{12}).$
Because all 12 eggs have the same mean weight, this becomes $\mathrm{E}(W) = \mathrm{E}(P) + 12 \times \mathrm{E}(X_i).$
We were told that $\mathrm{E}(W) = 840$ and $\mathrm{E}(P) = 20,$ so we can solve
$840 = 20 + 12 \times \mathrm{E}(X_i)$ to find $\mathrm{E}(X_i) = \dfrac{840 - 20}{12} \approx 68.33$ grams.

(ii) Because of independence, properties of variance establish that
$$\mathrm{Var}(W) = \mathrm{Var}(P) + \mathrm{Var}(X_1) + \mathrm{Var}(X_2) + \ldots + \mathrm{Var}(X_{12}).$$

Because all 12 eggs have the same variance of their weights, this becomes
$$\mathrm{Var}(W) = \mathrm{Var}(P) + 12 \times \mathrm{Var}(X_i).$$

We were told that $\mathrm{SD}(W) = 7.9$ and $\mathrm{SD}(P) = 1.7.$ Therefore, $\mathrm{Var}(W) = (7.9)^2 = 62.41$ and $\mathrm{Var}(P) = (1.7)^2 = 2.89.$

We can solve $62.41 = 2.89 + 12 \times \mathrm{Var}(X_i)$ to find $\mathrm{Var}(X_i) = \dfrac{62.41 - 2.89}{12} = 4.96.$ Thus, $\mathrm{SD}(X_i) = \sqrt{(4.96)} \approx 2.23$ grams.

Question 3 (continued)

Scoring

Parts (a), (b-i), and (b-ii) were scored as essentially correct (E), partially correct (P), or incorrect (I). (Minor arithmetic errors in any part were not penalized).

Part (a) is scored as follows:

Essentially correct (E) if the response correctly includes the following three components:

1. Indicates use of a normal distribution and clearly identifies the correct parameter values (using a z-score is sufficient);
2. Uses the correct boundary value;
3. Reports the correct normal probability consistent with components 1 and 2.

Partially correct (P) if the response correctly includes two of the three components listed above.

Incorrect (I) if the response does not satisfy the criteria for an E or a P.

Notes:

1. An error in statistical notation in the response lowers the score one level (that is, from E to P or from P to I).
2. Responses that calculate a probability for a sample mean with n not equal to 1 should be scored an I. For example, using $z = \frac{x - \mu}{\sigma/\sqrt{n}}$, even if the parameters were correctly identified.
3. In component 1, a sketch of a normal curve with the mean labeled is sufficient for indicating use of a normal distribution and identifying the mean.
4. The following were examples of clearly identified parameters for component 1:
   - Writes " $\mu = 840, \sigma = 7.9$."
   - Explicitly labels the mean and standard deviation in a normalcdf calculator statement.
   - Sketches a normal curve, labels 840 as the mean, and labels two additional consecutive values separated by 7.9.
5. For component 3, acceptable correct values were all in the interval from 0.1020 to 0.1038.

Part (b-i) is scored as follows:

Essentially correct (E) if the response correctly uses properties of expected values to set up the correct equation to be solved *AND* correctly solves the equation for the desired expected value
OR
If the response follows a correct numerical procedure to find the correct expected value for one egg.

Partially correct (P) if the response indicates a correct procedure but makes an error in applying properties of expected values.
OR
If the response provides poor communication of the procedure.

Question 3 (continued)

Incorrect (I) if the response does not satisfy the criteria for an E or P.

Note: $\frac{820}{12} = 68.33$ is an example of poor communication, because the two arithmetic steps of subtraction $(840 - 20)$ and division $\left(\frac{820}{12}\right)$ were not documented.

Part (b-ii) is scored as follows:

Essentially correct (E) if the response combines variances and correctly includes the following three components:

1. Subtracts variances
2. Correctly uses the "12" in the calculations
3. Reports the correct standard deviation, consistent with components (1) and (2)

Partially correct (P) if the response combines variances and correctly includes two of the three components listed above.

Incorrect (I) if the response does not satisfy the criteria for an E or P.

Notes:

1. Examples of incorrect calculations with variances that should be scored P (one component incorrect):

$$\sqrt{\frac{7.9^2 + 1.7^2}{12}} = 2.33\text{g}$$

$$\sqrt{7.9^2 - 1.7^2} = 7.71\text{g}$$

$$\frac{\sqrt{7.9^2 - 1.7^2}}{12} = 0.643\text{g}$$

$$\frac{7.9^2 - 1.7^2}{12} = 4.96\text{g}$$

Examples of incorrect calculations with variances that should be scored I (more than one component incorrect):

$$\sqrt{7.9^2 + 1.7^2} = 8.08\text{g}$$

$$\frac{\sqrt{7.9^2 + 1.7^2}}{12} = 0.673\text{g}$$

2. Example of a response that does not combine variances and should be scored I:

$$\sqrt{\frac{7.9^2}{12}} = 2.28$$

Question 3 (continued)

4 Complete Response

All three parts essentially correct

3 Substantial Response

Two parts essentially correct and one part partially correct

OR

One part essentially correct and two parts partially correct

2 Developing Response

Two parts essentially correct and one part incorrect

OR

One part essentially correct, one part partially correct, and one part incorrect

OR

Three parts partially correct

1 Minimal Response

One part essentially correct and two parts incorrect

OR

One or two parts partially correct

3. Each full carton of Grade A eggs consists of 1 randomly selected empty cardboard container and 12 randomly selected eggs. The weights of such full cartons are approximately normally distributed with a mean of 840 grams and a standard deviation of 7.9 grams.

(a) What is the probability that a randomly selected full carton of Grade A eggs will weigh more than 850 grams?

X_{full} = weight of a full carton

$X_{full} \sim N(840, 7.9)$

$P(X_{full} > 850) = \boxed{0.1028}$

(b) The weights of the empty cardboard containers have a mean of 20 grams and a standard deviation of 1.7 grams. It is reasonable to assume independence between the weights of the empty cardboard containers and the weights of the eggs. It is also reasonable to assume independence among the weights of the 12 eggs that are randomly selected for a full carton.

Let the random variable X be the weight of a single randomly selected Grade A egg.

i) What is the mean of X?

Y = weight of empty container

Z = weight of full carton

$Z = Y + X_1 + \cdots + X_{12}$

$\mu_Z = \mu_Y + 12\mu_X \Rightarrow \mu_X = \frac{\mu_Z - \mu_Y}{12} = \frac{840 - 20}{12}$

$= \boxed{68.333g}$

ii) What is the standard deviation of X?

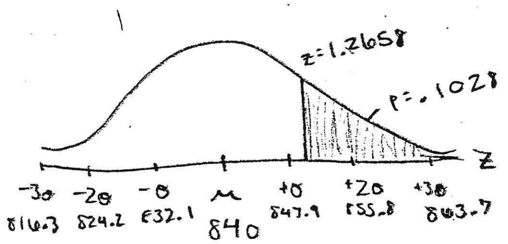
$Z = Y + X_1 + \cdots + X_{12}$

$\sigma_Z^2 = \sigma_Y^2 + 12\sigma_X^2$ (Since: $Y, X_1, X_2, \ldots, X_{12}$ independent)

$\sigma_X = \sqrt{\frac{\sigma_Z^2 - \sigma_Y^2}{12}} = \sqrt{\frac{7.9^2 - 1.7^2}{12}} = \boxed{2.2271g}$

3. Each full carton of Grade A eggs consists of 1 randomly selected empty cardboard container and 12 randomly selected eggs. The weights of such full cartons are approximately normally distributed with a mean of 840 grams and a standard deviation of 7.9 grams.

   (a) What is the probability that a randomly selected full carton of Grade A eggs will weigh more than 850 grams?



$z = \frac{x - \bar{x}}{\sigma} = \frac{850 - 840}{7.9} = 1.2658$

$P(z > 1.2658) = .102792$

$p = .1028$

   (b) The weights of the empty cardboard containers have a mean of 20 grams and a standard deviation of 1.7 grams. It is reasonable to assume independence between the weights of the empty cardboard containers and the weights of the eggs. It is also reasonable to assume independence among the weights of the 12 eggs that are randomly selected for a full carton.

   Let the random variable X be the weight of a single randomly selected Grade A egg.

   i) What is the mean of X?

Mean of full carton = 840g   Mean of empty carton = 20g

$\mu_x = \frac{840 - 20}{12} = 68.3333$ g

Mean of X = 68.3333 g

   ii) What is the standard deviation of X?

σ of full carton = 7.9 g → σ^2 of full carton = 62.41

σ of empty carton = 1.7 g → σ^2 of empty carton = 2.89

$\sigma_x^2 = \frac{62.41 - 2.89}{12} = 4.96$ g → $\sqrt{4.44} = 2.2271$

Standard deviation of x = 2.2271 g

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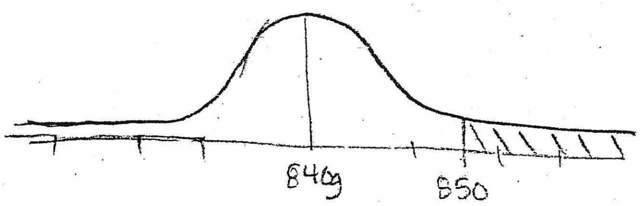
3C

3. Each full carton of Grade A eggs consists of 1 randomly selected empty cardboard container and 12 randomly selected eggs. The weights of such full cartons are approximately normally distributed with a mean of 840 grams and a standard deviation of 7.9 grams.

(a) What is the probability that a randomly selected full carton of Grade A eggs will weigh more than 850 grams?



$p = .1028$

(b) The weights of the empty cardboard containers have a mean of 20 grams and a standard deviation of 1.7 grams. It is reasonable to assume independence between the weights of the empty cardboard containers and the weights of the eggs. It is also reasonable to assume independence among the weights of the 12 eggs that are randomly selected for a full carton.

Let the random variable X be the weight of a single randomly selected Grade A egg.

i) What is the mean of X?

$\mu_{container} = 20g$

$\mu_{container + eggs} = 840g$

$\mu_x = (\mu_{C+E} - \mu_C) \div 12$

$= 68.3333\ g$

ii) What is the standard deviation of X?

$\sigma_{container} = 1.7g$

$\sigma_{container + eggs} = 7.9g$

$\sigma_x = \left(\sqrt{(1.7)^2 + (7.9g)^2}\right) \div 12$

$= 8.0808 \div 12$

$= .6734g$

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Question 3

Overview

The primary goals of this question were to assess a student's ability to (1) calculate a probability from a normal distribution and (2) apply properties of means and variances of functions of random variables.

Sample: 3A
Score: 4

In part (a) the response correctly indicates the use of the normal distribution and identifies the mean and standard deviation using standard statistical notation, satisfying the first component. The response indicates the correct boundary value and correct direction in the probability statement and arrives at the correct probability satisfying the second and third components. Because each of the three components was satisfied, part (a) was scored as essentially correct. In part (b-i) the response defines Y to be the weight of an empty container and Z to be the weight of a full carton. The response sets up a correct equation relating the weight of a full carton to the sum of the weights of its components and correctly uses the properties of expected values to set up an equation for the mean weight of a full carton. The response solves the equation for the requested mean, the mean weight of a single egg, substitutes the correct values, and reports the correct answer of 68.333 g. Because properties of expected values are used correctly, a correct equation is set up and solved, and the correct answer is reported, part (b-i) was scored as essentially correct. In part (b-ii) the response repeats the equation given in part (b-i), relating the weight of a full carton to the sum of the weights of its components and correctly uses the property of variances to set up a correct equation for the variance of weights of a full carton. An added comment, that the equation is valid since the random variables are independent, demonstrates understanding and adds strength to the response. The equation is correctly solved for the requested standard deviation, and the correct answer is reported, satisfying the third component. In the equation for the standard deviation of X the variances are subtracted, satisfying the first component, and the difference in variances is correctly divided by 12, satisfying the second component. With all three components satisfied, part (b-ii) was scored as essentially correct. Because three parts were scored as essentially correct, the response earned a score of 4.

Sample: 3B
Score: 3

In part (a) the response has a well labeled sketch of a normal distribution. The mean, as well as the values of three consecutive standard deviations above and below the mean, are clearly labeled. Additionally, a z-score equation with the correct parameter values substituted is given. Either one by itself is sufficient to satisfy the first component. The use of the correct boundary value, the second component, is also satisfied in two ways. It is correctly entered in the z-score, and on the sketch the boundary line is drawn, labeled ($z = 1.2658$) and the appropriate region is shaded. The correct normal probability calculation of 0.1028 is reported, satisfying the third component. All three components of part (a) are satisfied; however, in the initial z-score equation the population mean is incorrectly specified as a sample mean. With this error in statistical notation the response was scored as partially correct. The response in part (b-i) indentifies the values of the mean weight of a full carton and the mean weight of an empty carton and correctly enters these values into an equation for the mean of X. The equation is solved and the correct answer reported. Because properties of expected values are used correctly, a correct equation is set up and solved, and the correct answer is reported, part (b-i) was scored as essentially correct. The response in part (b-ii) indentifies the values of the standard deviations and variances of the weights of full cartons and the weights of empty cartons. A correct equation subtracting the variances and correctly dividing by 12 is given and solved. Thus the first two components are satisfied. The square root of the resulting variance is computed to arrive

Question 3 (continued)

at the correct value of the requested standard deviation, which satisfies the third component. Because each of the three components was satisfied, part (b-ii) was as scored essentially correct. Because two parts were scored as essentially correct and one part was scored as partially correct, the response earned a score of 3.

Sample: 3C
Score: 2

In part (a) the response has a sketch of a normal distribution with the mean (840) correctly labeled. This is sufficient to indicate the use of a normal distribution and for identifying the mean; however, the standard deviation is not identified, therefore the first component is not satisfied. The boundary value of 850 is clearly labeled on the sketch, the appropriate area is shaded, and the correct probability is reported. Therefore the second and third components are satisfied. With two of the three components satisfied, part (a) was scored as partially correct. In part (b-i) the response uses statistical notation to denote the values of the mean weight of the "container" and the mean weight of the "container + eggs." Properties of expected values are used correctly to set up an equation for the mean weight of a single egg. The equation is solved correctly, and the answer is reported. Because the properties of expected values are used correctly in setting up a correct equation, the equation is solved correctly, and the correct answer reported, part (b-i) was scored as essentially correct. In part (b-ii) the response combines variances; however, the response incorrectly adds the variances and incorrectly divides by 12 after taking the square root of the resulting variance. Thus neither of the first two components is satisfied. The standard deviation consistent with the calculated variance is computed satisfying the third component. With the first and second components incorrect and the third component satisfied, part (b-ii) was scored as incorrect. Because one part was scored as essentially correct one part was scored as partially correct, and one part was scored as incorrect, the response earned a score of 2.