AP® Calculus AB
1999 Scoring Guidelines

1. A particle moves along the y–axis with velocity given by $v(t) = t\sin(t^2)$ for $t \geq 0$.
   (a) In which direction (up or down) is the particle moving at time $t = 1.5$? Why?
   (b) Find the acceleration of the particle at time $t = 1.5$. Is the velocity of the particle increasing at $t = 1.5$? Why or why not?
   (c) Given that $y(t)$ is the position of the particle at time t and that $y(0) = 3$, find $y(2)$.
   (d) Find the total distance traveled by the particle from $t = 0$ to $t = 2$.

(a) $v(1.5) = 1.5\sin(1.5^2) = 1.167$

Up, because $v(1.5) > 0$

1: answer and reason

(b) $a(t) = v'(t) = \sin t^2 + 2t^2\cos t^2$

$a(1.5) = v'(1.5) = -2.048 \quad \text{or} \quad -2.049$

No; v is decreasing at 1.5 because $v'(1.5) < 0$

$$\mathbf{2}\begin{cases} 1\text{: } a(1.5) \\ 1\text{: conclusion and reason} \end{cases}$$

(c) $y(t) = \displaystyle\int v(t)\,dt$

$$= \int t\sin t^2\,dt = -\frac{\cos t^2}{2} + C$$

$$y(0) = 3 = -\frac{1}{2} + C \quad \Longrightarrow \quad C = \frac{7}{2}$$

$$y(t) = -\frac{1}{2}\cos t^2 + \frac{7}{2}$$

$$y(2) = -\frac{1}{2}\cos 4 + \frac{7}{2} = 3.826 \quad \text{or} \quad 3.827$$

$$\mathbf{3}\begin{cases} 1\text{: } y(t) = \displaystyle\int v(t)\,dt \\ 1\text{: } y(t) = -\dfrac{1}{2}\cos t^2 + C \\ 1\text{: } y(2) \end{cases}$$

(d) distance $= \displaystyle\int_0^2 |v(t)|\,dt = 1.173$

or

$v(t) = t\sin t^2 = 0$

$t = 0 \quad \text{or} \quad t = \sqrt{\pi} \approx 1.772$

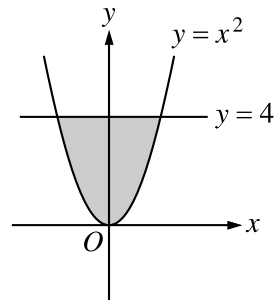
$y(0) = 3\,; \quad y(\sqrt{\pi}) = 4\,; \quad y(2) = 3.826$ or 3.827

$[y(\sqrt{\pi}) - y(0)] + [y(\sqrt{\pi}) - y(2)]$

$\quad = 1.173 \quad \text{or} \quad 1.174$

3 {
- 1: limits of 0 and 2 on an integral of $v(t)$ or $|v(t)|$
  or
  uses $y(0)$ and $y(2)$ to compute distance
- 1: handles change of direction at student's turning point
- 1: answer
  0/1 if incorrect turning point

2. The shaded region, R, is bounded by the graph of $y = x^2$ and the line $y = 4$, as shown in the figure above.
   (a) Find the area of R.
   (b) Find the volume of the solid generated by revolving R about the x–axis.
   (c) There exists a number k, $k > 4$, such that when R is revolved about the line $y = k$, the resulting solid has the same volume as the solid in part (b). Write, but do not solve, an equation involving an integral expression that can be used to find the value of k.



(a) Area $= \int_{-2}^{2} (4 - x^2)\,dx$

$$= 2\int_{0}^{2} (4 - x^2)\,dx$$

$$= 2\left[4x - \frac{x^3}{3}\right]_0^2$$

$$= \frac{32}{3} = 10.666 \quad \text{or} \quad 10.667$$

$2 \begin{cases} 1\text{: integral} \\ 1\text{: answer} \end{cases}$

(b) Volume $= \pi \int_{-2}^{2} \left(4^2 - (x^2)^2\right)\,dx$

$$= 2\pi \int_{0}^{2} (16 - x^4)\,dx$$

$$= 2\pi \left[16x - \frac{x^5}{5}\right]_0^2$$

$$= \frac{256\pi}{5} = 160.849 \quad \text{or} \quad 160.850$$

$3 \begin{cases} 1\text{: limits and constant} \\ 1\text{: integrand} \\ 1\text{: answer} \end{cases}$

(c) $\pi \int_{-2}^{2} \left[(k - x^2)^2 - (k-4)^2\right]\,dx = \frac{256\pi}{5}$

$4 \begin{cases} 1\text{: limits and constant} \\ 2\text{: integrand} \\ \quad <-1> \text{ each error} \\ 1\text{: equation} \end{cases}$

3. The rate at which water flows out of a pipe, in gallons per hour, is given by a differentiable function R of time t. The table above shows the rate as measured every 3 hours for a 24–hour period.

t (hours)	$R(t)$ (gallons per hour)
0	9.6
3	10.4
6	10.8
9	11.2
12	11.4
15	11.3
18	10.7
21	10.2
24	9.6

(a) Use a midpoint Riemann sum with 4 subdivisions of equal length to approximate $\int_0^{24} R(t)\,dt$. Using correct units, explain the meaning of your answer in terms of water flow.

(b) Is there some time t, $0 < t < 24$, such that $R'(t) = 0$? Justify your answer.

(c) The rate of water flow $R(t)$ can be approximated by $Q(t) = \frac{1}{79}\left(768 + 23t - t^2\right)$. Use $Q(t)$ to approximate the average rate of water flow during the 24–hour time period. Indicate units of measure.

(a) $$\int_0^{24} R(t)\,dt \approx 6[R(3) + R(9) + R(15) + R(21)]$$
$$= 6[10.4 + 11.2 + 11.3 + 10.2]$$
$$= 258.6 \text{ gallons}$$

This is an approximation to the total flow in gallons of water from the pipe in the 24–hour period.

$$\mathbf{3}\begin{cases} 1\text{: } R(3) + R(9) + R(15) + R(21) \\ 1\text{: answer} \\ 1\text{: explanation} \end{cases}$$

(b) Yes;

Since $R(0) = R(24) = 9.6$, the Mean Value Theorem guarantees that there is a t, $0 < t < 24$, such that $R'(t) = 0$.

$$\mathbf{2}\begin{cases} 1\text{: answer} \\ 1\text{: MVT or equivalent} \end{cases}$$

(c) Average rate of flow

$\approx$ average value of $Q(t)$

$$= \frac{1}{24}\int_0^{24} \frac{1}{79}(768 + 23t - t^2)\,dt$$

$$= 10.785 \text{ gal/hr} \quad \text{or} \quad 10.784 \text{ gal/hr}$$

$$\mathbf{3}\begin{cases} 1\text{: limits and average value constant} \\ 1\text{: } Q(t) \text{ as integrand} \\ 1\text{: answer} \end{cases}$$

(units) Gallons in part (a) and gallons/hr in part (c), or equivalent.

1: units

AB–4 1999

4. Suppose that the function f has a continuous second derivative for all x, and that $f(0) = 2$, $f'(0) = -3$, and $f''(0) = 0$. Let g be a function whose derivative is given by $g'(x) = e^{-2x}(3f(x) + 2f'(x))$ for all x.
   (a) Write an equation of the line tangent to the graph of f at the point where $x = 0$.
   (b) Is there sufficient information to determine whether or not the graph of f has a point of inflection when $x = 0$? Explain your answer.
   (c) Given that $g(0) = 4$, write an equation of the line tangent to the graph of g at the point where $x = 0$.
   (d) Show that $g''(x) = e^{-2x}(-6f(x) - f'(x) + 2f''(x))$. Does g have a local maximum at $x = 0$? Justify your answer.

(a) Slope at $x = 0$ is $f'(0) = -3$

At $x = 0$, $y = 2$

$y - 2 = -3(x - 0)$

1: equation

(b) No. Whether $f''(x)$ changes sign at $x = 0$ is unknown. The only given value of $f''(x)$ is $f''(0) = 0$.

$\mathbf{2}\begin{cases} \text{1: answer} \\ \text{1: explanation} \end{cases}$

(c) $g'(x) = e^{-2x}(3f(x) + 2f'(x))$

$$\begin{aligned} g'(0) &= e^0(3f(0) + 2f'(0)) \\ &= 3(2) + 2(-3) = 0 \end{aligned}$$

$y - 4 = 0(x - 0)$

$y = 4$

$\mathbf{2}\begin{cases} \text{1: } g'(0) \\ \text{1: equation} \end{cases}$

(d) $g'(x) = e^{-2x}(3f(x) + 2f'(x))$

$$\begin{aligned} g''(x) &= (-2e^{-2x})(3f(x) + 2f'(x)) \\ &\quad + e^{-2x}(3f'(x) + 2f''(x)) \\ &= e^{-2x}(-6f(x) - f'(x) + 2f''(x)) \end{aligned}$$

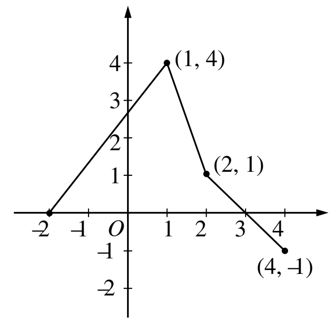
$g''(0) = e^0[(-6)(2) - (-3) + 2(0)] = -9$

Since $g'(0) = 0$ and $g''(0) < 0$, g does have a local maximum at $x = 0$.

$\mathbf{4}\begin{cases} \text{2: verify derivative} \\ \quad \text{0/2 product or chain rule error} \\ \quad \text{<−1> algebra errors} \\ \text{1: } g'(0) = 0 \text{ and } g''(0) \\ \text{1: answer and reasoning} \end{cases}$

AB–5 / BC–5 1999

5. The graph of the function f, consisting of three line segments, is given above. Let $g(x) = \int_1^x f(t)\,dt$.



(a) Compute $g(4)$ and $g(-2)$.

(b) Find the instantaneous rate of change of g, with respect to x, at $x = 1$.

(c) Find the absolute minimum value of g on the closed interval $[-2, 4]$. Justify your answer.

(d) The second derivative of g is not defined at $x = 1$ and $x = 2$. How many of these values are x–coordinates of points of inflection of the graph of g? Justify your answer.

(a) $g(4) = \int_1^4 f(t)\,dt = \frac{3}{2} + 1 + \frac{1}{2} - \frac{1}{2} = \frac{5}{2}$

$g(-2) = \int_1^{-2} f(t)\,dt = -\frac{1}{2}(12) = -6$

$$2 \begin{cases} 1\text{: } g(4) \\ 1\text{: } g(-2) \end{cases}$$

(b) $g'(1) = f(1) = 4$

1: answer

(c) g is increasing on $[-2, 3]$ and decreasing on $[3, 4]$.

Therefore, g has absolute minimum at an endpoint of $[-2, 4]$.

Since $g(-2) = -6$ and $g(4) = \frac{5}{2}$,

the absolute minimum value is -6.

$$3 \begin{cases} 1\text{: interior analysis} \\ 1\text{: endpoint analysis} \\ 1\text{: answer} \end{cases}$$

(d) One; $x = 1$

On $(-2, 1)$, $g''(x) = f'(x) > 0$

On $(1, 2)$, $g''(x) = f'(x) < 0$

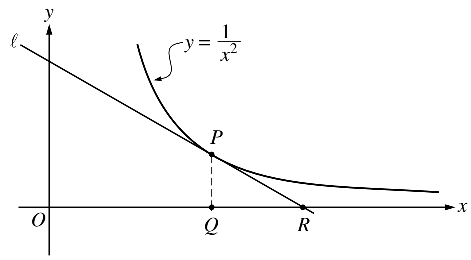
On $(2, 4)$, $g''(x) = f'(x) < 0$

Therefore $(1, g(1))$ is a point of inflection and $(2, g(2))$ is not.

$$3 \begin{cases} 1\text{: choice of } x = 1 \text{ only} \\ 1\text{: show } (1, g(1)) \text{ is a point of inflection} \\ 1\text{: show } (2, g(2)) \text{ is not a point of inflection} \end{cases}$$

AB–6 1999

6. In the figure above, line ℓ is tangent to the graph of $y = \dfrac{1}{x^2}$ at point P, with coordinates $\left(w, \dfrac{1}{w^2}\right)$, where $w > 0$. Point Q has coordinates $(w, 0)$. Line ℓ crosses the x–axis at the point R, with coordinates $(k, 0)$.



(a) Find the value of k when $w = 3$.

(b) For all $w > 0$, find k in terms of w.

(c) Suppose that w is increasing at the constant rate of 7 units per second. When $w = 5$, what is the rate of change of k with respect to time?

(d) Suppose that w is increasing at the constant rate of 7 units per second. When $w = 5$, what is the rate of change of the area of $\triangle PQR$ with respect to time? Determine whether the area is increasing or decreasing at this instant.

(a) $\dfrac{dy}{dx} = -\dfrac{2}{x^3}$; $\left.\dfrac{dy}{dx}\right|_{x=3} = -\dfrac{2}{27}$

Line ℓ through $\left(3, \dfrac{1}{9}\right)$ and $(k, 0)$ has slope $-\dfrac{2}{27}$.

Therefore, $\dfrac{0 - \frac{1}{9}}{k - 3} = -\dfrac{2}{27}$ or $0 - \dfrac{1}{9} = -\dfrac{2}{27}(k - 3)$

$k = \dfrac{9}{2}$

$$2\begin{cases} 1\text{: } \left.\dfrac{dy}{dx}\right|_{x=3} \\ 1\text{: answer} \end{cases}$$

(b) Line ℓ through $\left(w, \dfrac{1}{w^2}\right)$ and $(k, 0)$ has slope $-\dfrac{2}{w^3}$.

Therefore, $\dfrac{0 - \frac{1}{w^2}}{k - w} = -\dfrac{2}{w^3}$ or $0 - \dfrac{1}{w^2} = -\dfrac{2}{w^3}(k - w)$

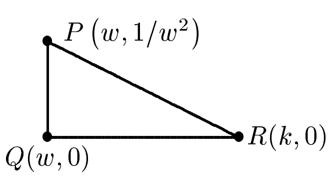
$k = \dfrac{3}{2}w$

$$2\begin{cases} 1\text{: equation relating } w \text{ and } k, \\ \quad\text{using slopes} \\ 1\text{: answer} \end{cases}$$

(c) $\dfrac{dk}{dt} = \dfrac{3}{2}\dfrac{dw}{dt} = \dfrac{3}{2} \cdot 7 = \dfrac{21}{2}$; $\left.\dfrac{dk}{dt}\right|_{w=5} = \dfrac{21}{2}$

1: answer using $\dfrac{dw}{dt} = 7$

(d)



$A = \dfrac{1}{2}(k - w)\dfrac{1}{w^2} = \dfrac{1}{2}\left(\dfrac{3}{2}w - w\right)\dfrac{1}{w^2} = \dfrac{1}{4w}$

$\dfrac{dA}{dt} = -\dfrac{1}{4w^2}\dfrac{dw}{dt}$

$\left.\dfrac{dA}{dt}\right|_{w=5} = -\dfrac{1}{100} \cdot 7 = -0.07$

Therefore, area is decreasing.

$$4\begin{cases} 1\text{: area in terms of } w \text{ and/or } k \\ 1\text{: } \dfrac{dA}{dt} \text{ implicitly} \\ 1\text{: } \left.\dfrac{dA}{dt}\right|_{w=5} \text{ using } \dfrac{dw}{dt} = 7 \\ 1\text{: conclusion} \end{cases}$$

Note: 0/4 if A constant