AP Calculus AB
2000 Scoring Guidelines

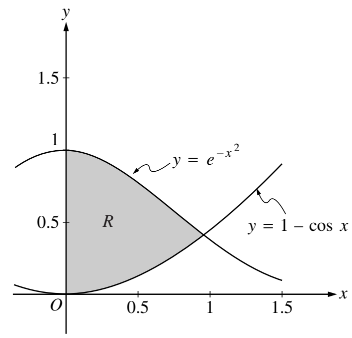
AP Calculus AB–1 / BC–1 2000

Let R be the shaded region in the first quadrant enclosed by the graphs of $y = e^{-x^2}$, $y = 1 - \cos x$, and the y-axis, as shown in the figure above.

(a) Find the area of the region R.

(b) Find the volume of the solid generated when the region R is revolved about the x-axis.

(c) The region R is the base of a solid. For this solid, each cross section perpendicular to the x-axis is a square. Find the volume of this solid.



Region R

$e^{-x^2} = 1 - \cos x$ at $x = 0.941944 = A$

1 : Correct limits in an integral in (a), (b), or (c).

(a) Area $= \int_0^A \left(e^{-x^2} - (1 - \cos x)\right) dx$

$= 0.590$ or 0.591

$2 \begin{cases} 1: \text{ integrand} \\ 1: \text{ answer} \end{cases}$

(b) Volume $= \pi \int_0^A \left(\left(e^{-x^2}\right)^2 - (1 - \cos x)^2 \right) dx$

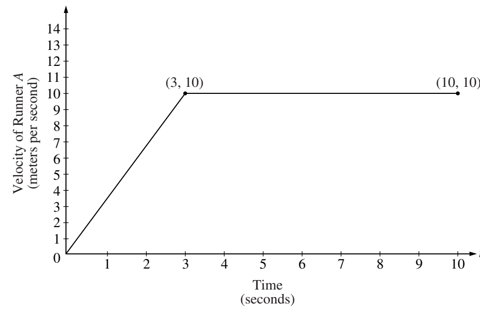
$= 0.55596\pi = 1.746$ or 1.747

$3 \begin{cases} 2: \text{ integrand and constant} \\ \quad <-1> \text{ each error} \\ 1: \text{ answer} \end{cases}$

(c) Volume $= \int_0^A \left(e^{-x^2} - (1 - \cos x)\right)^2 dx$

$= 0.461$

3
- 2 : integrand
  - $<-1>$ each error
  - Note: 0/2 if not of the form $k\int_c^d (f(x) - g(x))^2\, dx$
- 1 : answer



Two runners, A and B, run on a straight racetrack for $0 \le t \le 10$ seconds. The graph above, which consists of two line segments, shows the velocity, in meters per second, of Runner A. The velocity, in meters per second, of Runner B is given by the function v defined by $v(t) = \dfrac{24t}{2t+3}$.

(a) Find the velocity of Runner A and the velocity of Runner B at time $t = 2$ seconds. Indicate units of measure.

(b) Find the acceleration of Runner A and the acceleration of Runner B at time $t = 2$ seconds. Indicate units of measure.

(c) Find the total distance run by Runner A and the total distance run by Runner B over the time interval $0 \le t \le 10$ seconds. Indicate units of measure.

(a) Runner A: velocity $= \dfrac{10}{3} \cdot 2 = \dfrac{20}{3}$

$= 6.666$ or 6.667 meters/sec

Runner B: $v(2) = \dfrac{48}{7} = 6.857$ meters/sec

$2 \begin{cases} 1: \text{ velocity for Runner } A \\ 1: \text{ velocity for Runner } B \end{cases}$

(b) Runner A: acceleration $= \dfrac{10}{3} = 3.333$ meters/sec^2

Runner B: $a(2) = v'(2) = \left.\dfrac{72}{(2t+3)^2}\right|_{t=2}$

$= \dfrac{72}{49} = 1.469$ meters/sec^2

$2 \begin{cases} 1: \text{ acceleration for Runner } A \\ 1: \text{ acceleration for Runner } B \end{cases}$

(c) Runner A: distance $= \dfrac{1}{2}(3)(10) + 7(10) = 85$ meters

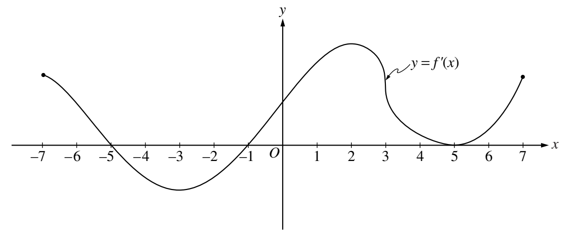
Runner B: distance $= \displaystyle\int_0^{10} \frac{24t}{2t+3}\,dt = 83.336$ meters

$4 \begin{cases} 2: \text{ distance for Runner } A \\ \quad 1: \text{ method} \\ \quad 1: \text{ answer} \\ 2: \text{ distance for Runner } B \\ \quad 1: \text{ integral} \\ \quad 1: \text{ answer} \end{cases}$

(units) meters/sec in part (a), meters/sec^2 in part (b), and meters in part (c), or equivalent.

1: units

AP Calculus AB–3 2000



The figure above shows the graph of f', the derivative of the function f, for $-7 \le x \le 7$. The graph of f' has horizontal tangent lines at $x = -3$, $x = 2$, and $x = 5$, and a vertical tangent line at $x = 3$.

(a) Find all values of x, for $-7 < x < 7$, at which f attains a relative minimum. Justify your answer.

(b) Find all values of x, for $-7 < x < 7$, at which f attains a relative maximum. Justify your answer.

(c) Find all values of x, for $-7 < x < 7$, at which $f''(x) < 0$.

(d) At what value of x, for $-7 \le x \le 7$, does f attain its absolute maximum? Justify your answer.

(a) $x = -1$

$f'(x)$ changes from negative to positive at $x = -1$

2 { 1 : answer; 1 : justification }

(b) $x = -5$

$f'(x)$ changes from positive to negative at $x = -5$

2 { 1 : answer; 1 : justification }

(c) $f''(x)$ exists and f' is decreasing on the intervals $(-7, -3)$, $(2, 3)$, and $(3, 5)$

2 { 1 : $(-7, -3)$; 1 : $(2, 3) \cup (3, 5)$ }

(d) $x = 7$

The absolute maximum must occur at $x = -5$ or at an endpoint.

$f(-5) > f(-7)$ because f is increasing on $(-7, -5)$

The graph of f' shows that the magnitude of the negative change in f from $x = -5$ to $x = -1$ is smaller than the positive change in f from $x = -1$ to $x = 7$. Therefore the net change in f is positive from $x = -5$ to $x = 7$, and $f(7) > f(-5)$. So $f(7)$ is the absolute maximum.

3 {
- 1 : answer
- 1 : identifies $x = -5$ and $x = 7$ as candidates
  — or —
  indicates that the graph of f increases, decreases, then increases
- 1 : justifies $f(7) > f(-5)$

}

Water is pumped into an underground tank at a constant rate of 8 gallons per minute. Water leaks out of the tank at the rate of $\sqrt{t+1}$ gallons per minute, for $0 \le t \le 120$ minutes. At time $t = 0$, the tank contains 30 gallons of water.

(a) How many gallons of water leak out of the tank from time $t = 0$ to $t = 3$ minutes?

(b) How many gallons of water are in the tank at time $t = 3$ minutes?

(c) Write an expression for $A(t)$, the total number of gallons of water in the tank at time t.

(d) At what time t, for $0 \le t \le 120$, is the amount of water in the tank a maximum? Justify your answer.

(a) Method 1: $\int_0^3 \sqrt{t+1}\,dt = \frac{2}{3}(t+1)^{3/2}\Big|_0^3 = \frac{14}{3}$

– or –

Method 2: $L(t)$ = gallons leaked in first t minutes

$$\frac{dL}{dt} = \sqrt{t+1}; \quad L(t) = \frac{2}{3}(t+1)^{3/2} + C$$

$$L(0) = 0; \quad C = -\frac{2}{3}$$

$$L(t) = \frac{2}{3}(t+1)^{3/2} - \frac{2}{3}; \quad L(3) = \frac{14}{3}$$

Scoring:

Method 1:

3 { 2 : definite integral { 1 : limits; 1 : integrand }; 1 : answer }

– or –

Method 2:

3 { 1 : antiderivative with C; 1 : solves for C using $L(0) = 0$; 1 : answer }

(b) $30 + 8 \cdot 3 - \frac{14}{3} = \frac{148}{3}$

Scoring: 1 : answer

(c) Method 1:

$$A(t) = 30 + \int_0^t \left(8 - \sqrt{x+1}\right)dx$$

$$= 30 + 8t - \int_0^t \sqrt{x+1}\,dx$$

– or –

Method 2:

$$\frac{dA}{dt} = 8 - \sqrt{t+1}$$

$$A(t) = 8t - \frac{2}{3}(t+1)^{3/2} + C$$

$$30 = 8(0) - \frac{2}{3}(0+1)^{3/2} + C; \quad C = \frac{92}{3}$$

$$A(t) = 8t - \frac{2}{3}(t+1)^{3/2} + \frac{92}{3}$$

Scoring:

Method 1:

2 { 1 : $30 + 8t$; 1 : $-\int_0^t \sqrt{x+1}\,dx$ }

– or –

Method 2:

2 { 1 : antiderivative with C; 1 : answer }

(d) $A'(t) = 8 - \sqrt{t+1} = 0$ when $t = 63$

$A'(t)$ is positive for $0 < t < 63$ and negative for $63 < t < 120$. Therefore there is a maximum at $t = 63$.

Scoring:

3 { 1 : sets $A'(t) = 0$; 1 : solves for t; 1 : justification }

Consider the curve given by $xy^2 - x^3y = 6$.

(a) Show that $\frac{dy}{dx} = \frac{3x^2y - y^2}{2xy - x^3}$.

(b) Find all points on the curve whose x-coordinate is 1, and write an equation for the tangent line at each of these points.

(c) Find the x-coordinate of each point on the curve where the tangent line is vertical.

(a) $y^2 + 2xy\frac{dy}{dx} - 3x^2y - x^3\frac{dy}{dx} = 0$

$\frac{dy}{dx}\left(2xy - x^3\right) = 3x^2y - y^2$

$\frac{dy}{dx} = \frac{3x^2y - y^2}{2xy - x^3}$

2 { 1 : implicit differentiation; 1 : verifies expression for $\frac{dy}{dx}$ }

(b) When $x = 1$, $y^2 - y = 6$

$y^2 - y - 6 = 0$

$(y-3)(y+2) = 0$

$y = 3,\ y = -2$

At $(1,3)$, $\frac{dy}{dx} = \frac{9-9}{6-1} = 0$

Tangent line equation is $y = 3$

At $(1,-2)$, $\frac{dy}{dx} = \frac{-6-4}{-4-1} = \frac{-10}{-5} = 2$

Tangent line equation is $y + 2 = 2(x-1)$

4 { 1 : $y^2 - y = 6$; 1 : solves for y; 2 : tangent lines }

Note: 0/4 if not solving an equation of the form $y^2 - y = k$

(c) Tangent line is vertical when $2xy - x^3 = 0$

$x\left(2y - x^2\right) = 0$ gives $x = 0$ or $y = \frac{1}{2}x^2$

There is no point on the curve with x-coordinate 0.

When $y = \frac{1}{2}x^2$, $\frac{1}{4}x^5 - \frac{1}{2}x^5 = 6$

$-\frac{1}{4}x^5 = 6$

$x = \sqrt[5]{-24}$

3 { 1 : sets denominator of $\frac{dy}{dx}$ equal to 0; 1 : substitutes $y = \frac{1}{2}x^2$ or $x = \pm\sqrt{2y}$ into the equation for the curve; 1 : solves for x-coordinate }

Consider the differential equation $\frac{dy}{dx} = \frac{3x^2}{e^{2y}}$.

(a) Find a solution $y = f(x)$ to the differential equation satisfying $f(0) = \frac{1}{2}$.

(b) Find the domain and range of the function f found in part (a).

(a) $e^{2y}\,dy = 3x^2\,dx$

$$\frac{1}{2}e^{2y} = x^3 + C_1$$

$$e^{2y} = 2x^3 + C$$

$$y = \frac{1}{2}\ln\left(2x^3 + C\right)$$

$$\frac{1}{2} = \frac{1}{2}\ln(0 + C); \quad C = e$$

$$y = \frac{1}{2}\ln(2x^3 + e)$$

6 {
- 1 : separates variables
- 1 : antiderivative of dy term
- 1 : antiderivative of dx term
- 1 : constant of integration
- 1 : uses initial condition $f(0) = \frac{1}{2}$
- 1 : solves for y

Note: 0/1 if y is not a logarithmic function of x

Note: max 3/6 [1-1-1-0-0-0] if no constant of integration

Note: 0/6 if no separation of variables

(b) Domain: $2x^3 + e > 0$

$$x^3 > -\frac{1}{2}e$$

$$x > \left(-\frac{1}{2}e\right)^{1/3} = -\left(\frac{1}{2}e\right)^{1/3}$$

Range: $-\infty < y < \infty$

3 {
- 1 : $2x^3 + e > 0$
- 1 : domain

  Note: 0/1 if 0 is not in the domain
- 1 : range

Note: 0/3 if y is not a logarithmic function of x