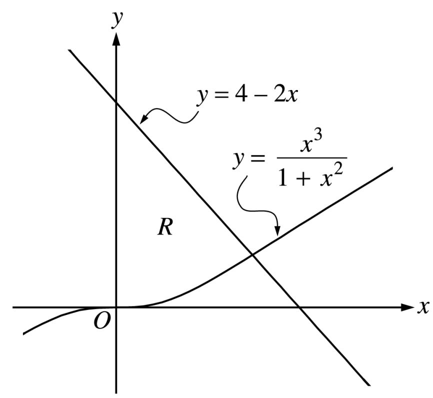
AP® Calculus AB
2002 Scoring Guidelines
Form B

Question 1

Let R be the region bounded by the y-axis and the graphs of $y = \dfrac{x^3}{1+x^2}$ and $y = 4 - 2x$, as shown in the figure above.



(a) Find the area of R.

(b) Find the volume of the solid generated when R is revolved about the x-axis.

(c) The region R is the base of a solid. For this solid, each cross section perpendicular to the x-axis is a square. Find the volume of this solid.

Region R

$$\frac{x^3}{1+x^2} = 4 - 2x \text{ at } x = 1.487664 = A$$

1 : Correct limits in an integral in (a), (b), or (c).

(a) Area $= \displaystyle\int_0^A \left(4 - 2x - \frac{x^3}{1+x^2}\right) dx$

$= 3.214$ or 3.215

2 { 1 : integrand; 1 : answer }

(b) Volume

$$= \pi \int_0^A \left((4-2x)^2 - \left(\frac{x^3}{1+x^2}\right)^2 \right) dx$$

$= 31.884$ or 31.885 or 10.149π

3 { 2 : integrand and constant; $< -1 >$ each error; 1 : answer }

(c) Volume $= \displaystyle\int_0^A \left(4 - 2x - \frac{x^3}{1+x^2}\right)^2 dx$

$= 8.997$

3 { 2 : integrand; $< -1 >$ each error; note: 0/2 if not of the form $k\displaystyle\int_c^d (f(x) - g(x))^2\, dx$; 1 : answer }

Question 2

The number of gallons, $P(t)$, of a pollutant in a lake changes at the rate $P'(t) = 1 - 3e^{-0.2\sqrt{t}}$ gallons per day, where t is measured in days. There are 50 gallons of the pollutant in the lake at time $t = 0$. The lake is considered to be safe when it contains 40 gallons or less of pollutant.

(a) Is the amount of pollutant increasing at time $t = 9$? Why or why not?

(b) For what value of t will the number of gallons of pollutant be at its minimum? Justify your answer.

(c) Is the lake safe when the number of gallons of pollutant is at its minimum? Justify your answer.

(d) An investigator uses the tangent line approximation to $P(t)$ at $t = 0$ as a model for the amount of pollutant in the lake. At what time t does this model predict that the lake becomes safe?

Solution	Scoring
(a) $P'(9) = 1 - 3e^{-0.6} = -0.646 < 0$ so the amount is not increasing at this time.	1 : answer with reason
(b) $P'(t) = 1 - 3e^{-0.2\sqrt{t}} = 0$ $t = (5\ln 3)^2 = 30.174$ $P'(t)$ is negative for $0 < t < (5\ln 3)^2$ and positive for $t > (5\ln 3)^2$. Therefore there is a minimum at $t = (5\ln 3)^2$.	3 { 1 : sets $P'(t) = 0$ 1 : solves for t 1 : justification
(c) $P(30.174) = 50 + \int_0^{30.174} \left(1 - 3e^{-0.2\sqrt{t}}\right) dt$ $= 35.104 < 40$, so the lake is safe.	3 { 1 : integrand 1 : limits 1 : conclusion with reason based on integral of $P'(t)$
(d) $P'(0) = 1 - 3 = -2$. The lake will become safe when the amount decreases by 10. A linear model predicts this will happen when $t = 5$.	2 { 1 : slope of tangent line 1 : answer

Question 3

A particle moves along the x-axis so that its velocity v at any time t, for $0 \le t \le 16$, is given by $v(t) = e^{2\sin t} - 1$. At time $t = 0$, the particle is at the origin.

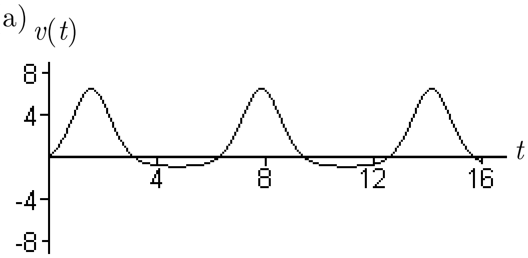
(a) On the axes provided, sketch the graph of $v(t)$ for $0 \le t \le 16$.

(b) During what intervals of time is the particle moving to the left? Give a reason for your answer.

(c) Find the total distance traveled by the particle from $t = 0$ to $t = 4$.

(d) Is there any time t, $0 < t \le 16$, at which the particle returns to the origin? Justify your answer.

(a)



1 : graph
- three "humps"
- periodic behavior
- starts at origin
- reasonable relative max and min values

(b) Particle is moving to the left when $v(t) < 0$, i.e. $e^{2\sin t} < 1$.

$(\pi, 2\pi)$, $(3\pi, 4\pi)$ and $(5\pi, 16]$

3 $\begin{cases} 2 : \text{intervals} \\ \quad < -1 > \text{ each missing or incorrect interval} \\ 1 : \text{reason} \end{cases}$

(c) $\int_0^4 |v(t)|\,dt = 10.542$

or

$v(t) = e^{2\sin t} - 1 = 0$

$t = 0$ or $t = \pi$

$x(\pi) = \int_0^{\pi} v(t)\,dt = 10.10656$

$x(4) = \int_0^{4} v(t)\,dt = 9.67066$

$|x(\pi) - x(0)| + |x(4) - x(\pi)|$

$= 10.542$

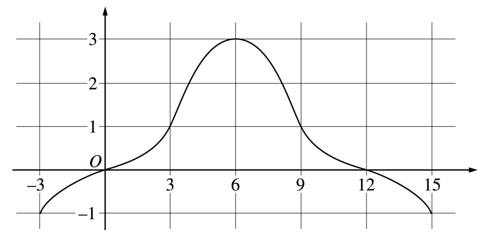
3 $\begin{cases} 1 : \text{limits of 0 and 4 on an integral of } v(t) \text{ or } |v(t)| \\ \quad \text{or} \\ \quad \text{uses } x(0) \text{ and } x(4) \text{ to compute distance} \\ 1 : \text{handles change of direction at student's turning point} \\ 1 : \text{answer} \\ \quad \text{note: 0/1 if incorrect turning point} \end{cases}$

(d) There is no such time because $\int_0^T v(t)\,dt > 0$ for all $T > 0$.

2 $\begin{cases} 1 : \text{no such time} \\ 1 : \text{reason} \end{cases}$

Question 4

The graph of a differentiable function f on the closed interval $[-3, 15]$ is shown in the figure above. The graph of f has a horizontal tangent line at $x = 6$. Let $g(x) = 5 + \int_6^x f(t)\,dt$ for $-3 \le x \le 15$.



Graph of f

(a) Find $g(6)$, $g'(6)$, and $g''(6)$.

(b) On what intervals is g decreasing? Justify your answer.

(c) On what intervals is the graph of g concave down? Justify your answer.

(d) Find a trapezoidal approximation of $\int_{-3}^{15} f(t)\,dt$ using six subintervals of length $\Delta t = 3$.

(a) $g(6) = 5 + \int_6^6 f(t)\,dt = 5$

$g'(6) = f(6) = 3$

$g''(6) = f'(6) = 0$

3 { 1 : $g(6)$; 1 : $g'(6)$; 1 : $g''(6)$ }

(b) g is decreasing on $[-3, 0]$ and $[12, 15]$ since $g'(x) = f(x) < 0$ for $x < 0$ and $x > 12$.

3 { 1 : $[-3, 0]$; 1 : $[12, 15]$; 1 : justification }

(c) The graph of g is concave down on $(6, 15)$ since $g' = f$ is decreasing on this interval.

2 { 1 : interval; 1 : justification }

(d) $\frac{3}{2}(-1 + 2(0 + 1 + 3 + 1 + 0) - 1)$
$= 12$

1 : trapezoidal method

Question 5

Consider the differential equation $\dfrac{dy}{dx} = \dfrac{3-x}{y}$.

(a) Let $y = f(x)$ be the particular solution to the given differential equation for $1 < x < 5$ such that the line $y = -2$ is tangent to the graph of *f*. Find the *x*-coordinate of the point of tangency, and determine whether *f* has a local maximum, local minimum, or neither at this point. Justify your answer.

(b) Let $y = g(x)$ be the particular solution to the given differential equation for $-2 < x < 8$, with the initial condition $g(6) = -4$. Find $y = g(x)$.

(a) $\dfrac{dy}{dx} = 0$ when $x = 3$

$$\left.\frac{d^2y}{dx^2}\right|_{(3,-2)} = \left.\frac{-y - y'(3-x)}{y^2}\right|_{(3,-2)} = \frac{1}{2},$$

so *f* has a local minimum at this point.

or

Because *f* is continuous for $1 < x < 5$, there is an interval containing $x = 3$ on which $y < 0$. On this interval, $\dfrac{dy}{dx}$ is negative to the left of $x = 3$ and $\dfrac{dy}{dx}$ is positive to the right of $x = 3$. Therefore *f* has a local minimum at $x = 3$.

3 {
- 1 : $x = 3$
- 1 : local minimum
- 1 : justification

(b) $y\,dy = (3-x)\,dx$

$\frac{1}{2}y^2 = 3x - \frac{1}{2}x^2 + C$

$8 = 18 - 18 + C$; $C = 8$

$y^2 = 6x - x^2 + 16$

$y = -\sqrt{6x - x^2 + 16}$

6 {
- 1 : separates variables
- 1 : antiderivative of dy term
- 1 : antiderivative of dx term
- 1 : constant of integration
- 1 : uses initial condition $g(6) = -4$
- 1 : solves for y

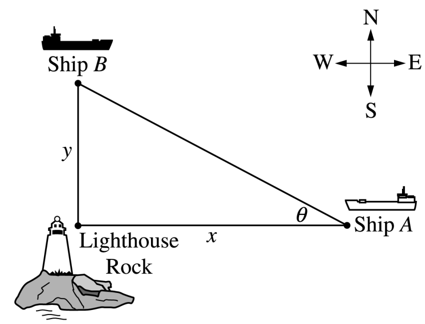
Note: max 3/6 [1-1-1-0-0-0] if no constant of integration

Note: 0/6 if no separation of variables

AP® CALCULUS AB
2002 SCORING GUIDELINES (Form B)

Question 6

Ship A is traveling due west toward Lighthouse Rock at a speed of 15 kilometers per hour (km/hr). Ship B is traveling due north away from Lighthouse Rock at a speed of 10 km/hr. Let x be the distance between Ship A and Lighthouse Rock at time t, and let y be the distance between Ship B and Lighthouse Rock at time t, as shown in the figure above.



(a) Find the distance, in kilometers, between Ship A and Ship B when $x = 4$ km and $y = 3$ km.

(b) Find the rate of change, in km/hr, of the distance between the two ships when $x = 4$ km and $y = 3$ km.

(c) Let θ be the angle shown in the figure. Find the rate of change of θ, in radians per hour, when $x = 4$ km and $y = 3$ km.

(a) Distance $= \sqrt{3^2 + 4^2} = 5$ km

1 : answer

(b) $r^2 = x^2 + y^2$

$$2r\frac{dr}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$$

or explicitly:

$$r = \sqrt{x^2 + y^2}$$

$$\frac{dr}{dt} = \frac{1}{2\sqrt{x^2 + y^2}}\left(2x\frac{dx}{dt} + 2y\frac{dy}{dt}\right)$$

At $x = 4,\ y = 3$,

$$\frac{dr}{dt} = \frac{4(-15) + 3(10)}{5} = -6 \text{ km/hr}$$

4 {
- 1 : expression for distance
- 2 : differentiation with respect to t
  - $< -2 >$ chain rule error
- 1 : evaluation

(c) $\tan\theta = \dfrac{y}{x}$

$$\sec^2\theta\,\frac{d\theta}{dt} = \frac{\frac{dy}{dt}x - \frac{dx}{dt}y}{x^2}$$

At $x = 4$ and $y = 3,\ \sec\theta = \dfrac{5}{4}$

$$\frac{d\theta}{dt} = \frac{16}{25}\left(\frac{10(4) - (-15)(3)}{16}\right)$$

$$= \frac{85}{25} = \frac{17}{5} \text{ radians/hr}$$

4 {
- 1 : expression for θ in terms of x and y
- 2 : differentiation with respect to t
  - $< -2 >$ chain rule, quotient rule, or transcendental function error
  - note: 0/2 if no trig or inverse trig function
- 1 : evaluation